# MATH 251 B45/B46 Quiz #4 Winter 2005

**Name:** K E Y **I.D.#:**

**Answer all questions. Calculators are NOT allowed. 30 minutes.**

1. Simplify: [10]

   (a) $$\log_2 16 = \log_2(2^4) = 4.$$

   (b) $$\frac{\log_3 0.01}{\log_3 10} = \log_{10} 0.01 = \log_{10}(10^{-2}) = -2.$$

2. Differentiate: [20]

   (*a*) $$y = (\cos x)^x, \qquad \ln y = x\ln(\cos x),$$
   $$\frac{y'}{y} = \ln(\cos x) + x\frac{1}{\cos x}(-\sin x) = \ln(\cos x) - x\tan x,$$
   $$y' = y(\ln(\cos x) - x\tan x) = (\cos x)^x(\ln(\cos x) - x\tan x)$$

   (*b*) $$y = x^2 \arcsin x, \qquad y' = 2x\arcsin x + x^2\frac{1}{\sqrt{1-x^2}} = 2x\arcsin x + \frac{x^2}{\sqrt{1-x^2}}.$$

3. A rectangle has two vertices on the $x$-axis and the other two on the parabola $y = 9 - x^2$. Determine its maximal area. [20]

**Solution:** The rectangle has width $2x$ and height $y = 9 - x^2$, so the area equals
$$A(x) = 2xy = 2x(9-x^2) = 2(9x - x^3).$$
We want to minimize the function $A(x)$ on the interval $x \in [0,3]$. Since the function $A$ is a polynomial, it is differentiable, so it has **no singular points**. In order to find critical points of $A$, we must solve the equation $A'(x) = 0$ for $x$:
$$A'(x) = 2(9-3x^2) = 6(3-x^2) = 6(\sqrt{3}-x)(\sqrt{3}+x),$$
so the only **critical point** is $x = \sqrt{3}$ (since $0 < \sqrt{3} < 3$, and the point $x = -\sqrt{3}$ does not belong to the domain). Let us find the **type** of this critical point.

The **1st derivative test:**

| $x$ | $(0,\sqrt{3})$ | $(\sqrt{3},3)$ |
|---|---|---|
| $A'(x)$ | $+$ | $-$ |
| $A$ | $\nearrow$ | $\searrow$ |

$\Longrightarrow$ **local max at $x = \sqrt{3}$**

**OR** The **2nd derivative test:** $A''(x) = -12x$, $A''(\sqrt{3}) = -12\sqrt{3} < 0$, so the function $A$ is **concave down**, so **local maximum**.

Thus, $A$ has an **absolute maximum** at the critical point $x = \sqrt{3}$.
The maximal area equals $A_{max} = A(\sqrt{3}) = 2(9\sqrt{3} - 3\sqrt{3}) = 12\sqrt{3}$.

**Answer:** $A_{max} = 12\sqrt{3}$.

4. Sketch the graph of $y = f(x) = \frac{\ln x}{x}$. [30]

(a) Find the domain, range, asymptotes.

(b) Find the intervals where $f$ is increasing or decreasing.

(c) Find the intervals where $f$ is concave up or concave down.

(d) Find the $(x, y)$-coordinates of all local minima, local maxima, inflexion points, $x$- and $y$-intercepts, and indicate them on the graph.

**Solution:** (a) **Domain:** $D_f = (0, \infty)$, since $\ln x$ is defined for $x > 0$ only.
**Vertical asymptote** can be $x = 0$ only, since $f$ is continuous everywhere on its domain. Test:

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} = \frac{\ln x}{x} = \text{“}\frac{-\infty}{0^+}\text{”} = (-\infty)\infty = -\infty,$$

so $x = 0$ is really the **vertical asymptote such that $y \to -\infty$ as $x \to 0^+$**. (Thus the graph approaches this asymptote from right as $y \to -\infty$.)

**Horizontal asymptote:**

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\ln x}{x} = \text{“}\frac{\infty}{\infty}\text{”} = \text{“}\frac{1}{\infty}\text{”} = 0^+,$$

since “in a battle between a logarithm and a polynomial, the polynomial always wins”; so $y = 0$ is a **horizontal asymptote such that $y \to 0^+$ as $x \to \infty$**. (Thus the graph approaches this asymptote from above as $x \to \infty$.)

**Slant asymptotes:** none.

**Now draw a small part of the graph near each asymptote!**

(b)
$$f'(x) = \frac{\frac{1}{x}x - \ln x}{x^2} = \frac{1 - \ln x}{x^2},$$

the **1st derivative test**:

| $x$ | $(0, e)$ | $(e, \infty)$ |
|---|---|---|
| $f'(x)$ | $+$ | $-$ |
| $f$ | ↗ | ↘ |

$\Longrightarrow$ **local max at $x = e$**

(c)
$$f''(x) = \frac{-\frac{1}{x}x^2 - 2x(1 - \ln x)}{x^4} = \frac{-x - 2x + 2x\ln x}{x^4} = \frac{-3 + 2\ln x}{x^3},$$

the **2nd derivative test**:

| $x$ | $(0, e^{3/2})$ | $(e^{3/2}, \infty)$ |
|---|---|---|
| $f''(x)$ | $-$ | $+$ |
| $f$ | ⌢ | ⌣ |

$\Longrightarrow$ **inflexion point at $x = e^{3/2}$**

(d) **local minima:** none, **local maximum:** $(e, \frac{1}{e})$ (by part (b));
**inflexion point:** $(e^{3/2}, \frac{3}{2e^{3/2}})$ (by part (c));
**$x$-intercept:** $(1, 0)$ (by solving equation $f(x) = 0$ for $x$);
**$y$-intercepts:** none (since $x = 0$ does not belong the domain).

**Sketching the graph:** Indicate the intercepts and the points of local maxima and minima, and join them, by monotone curves, with the parts of the graph (drawn in part (a)) near the asymptotes. **Make sure that no additional intercepts appear, and the intervals where $f$ is increasing or decreasing are exactly those found in part (b)!** Indicate the inflexion points on the graph.

(a) **Range:** By part (b), $f$ is increasing on $(0, e]$ from $-\infty$ up to $\frac{1}{e}$, and decreasing on $[e, \infty)$ from $\frac{1}{e}$ up to 0 (see also part (a)). So, the range is $R_f = (-\infty, \frac{1}{e}]$. (This can also be seen from the graph.)

5. (a) Approximate $\sqrt{27}$ using linearization. [20]

(b) Determine the sign of the Error, and estimate its size.
Is this estimate an overestimate or underestimate?

**Solution:** Since $\sqrt{25} = 5$, it is useful to approximate the function $y = f(x) = \sqrt{x}$ near the point $a = 25$. Let us approximate the graph of $y = f(x)$ by its tangent line at the point $(a, f(a)) = (25, 5)$. We need two derivatives of $f$:

$$f(x) = \sqrt{x}, \qquad f'(x) = \frac{1}{2\sqrt{x}}, \qquad f''(x) = -\frac{1}{4\sqrt{x^3}}.$$

The tangent line at the point $(25, 5)$ has slope $m = f'(25) = \frac{1}{10}$. The equation of the line having slope $m = \frac{1}{10}$ and going through the point $(25, 5)$ is

$$y = 5 + m \cdot (x - 25), \qquad y = 5 + \frac{x - 25}{10}.$$

The function $L(x) = 5 + \frac{x-25}{10}$ on the right hand side of this equation is called the **linearization of $f$ at the point** $a$. We have

$$\sqrt{27} = f(27) \approx L(27) = 5 + \frac{2}{10} = 5.2$$

(b) The Error $E(x) = f(x) - L(x)$ of the linear approximation has the following form:

$$E(x) = \frac{f''(c)}{2}(x - a)^2 \qquad \text{for some } c \in (a, x).$$

(Here we used that $a = 25 < 27 = x$.) Therefore

$$E(27) = \frac{f''(c)}{2}2^2 = -\frac{1}{2c^{3/2}} \qquad \text{for some } c \in (25, 27),$$

so $E(27) < 0$. Since (ERROR)=(TRUE VALUE)−(APPROXIMATION)$< 0$, our approximation 5.2 is an **overestimate** for $\sqrt{27}$.

Let us estimate the **absolute value of the Error**, using the inequality $25 < c < 27$:

$$|E(27)| = \frac{1}{2c^{3/2}} \quad < \quad \frac{1}{2 \cdot 25^{3/2}} = \frac{1}{2 \cdot 5^3} = \frac{1}{250} = 0.004$$

**Answer:** $\sqrt{27} \approx 5.2$, $E(27) < 0$, $|E(27)| < .004$, an overestimate.