MATH 251 B45/B46 Quiz #5 Winter 2005

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Answer all questions. Calculators are NOT allowed. 30 minutes.

1. Calculate the Taylor polynomial of degree 4 for $f(x) = \cos x$ about $x = 0$. [15]

Solution: We need 4 derivatives of f at $x = 0$:

$$f'(x) = -\sin x, \qquad f''(x) = -\cos x, \qquad f'''(x) = \sin x, \qquad f''''(x) = \cos x,$$

now evaluate at $x = 0$:

$$f(0) = 1, \qquad f'(0) = 0, \qquad f''(0) = -1, \qquad f'''(0) = 0, \qquad f''''(0) = 1,$$

so

$$P_4(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \frac{f''''(0)}{4!}(x-0)^4$$

$$= 1 + 0x + \frac{-1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4 = 1 - \frac{x^2}{2} + \frac{x^4}{24}.$$

Answer: $P_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$.

2. Evaluate [20]

$$\lim_{x\to 1} \frac{x \ln x}{x^3 - x^2} = \lim_{x\to 1} \frac{\ln x}{x^2 - x} = \text{“}\frac{0}{1-1}\text{”} = \text{“}\frac{0}{0}\text{”} = \text{(by l'Hôpital's Rule)}$$

$$= \lim_{x\to 1} \frac{(\ln x)'}{(x^2 - x)'} = \lim_{x\to 1} \frac{1/x}{2x - 1} = \lim_{x\to 1} \frac{1}{x(2x-1)} == \frac{1}{1(2-1)} = 1.$$

Answer: 1.

3. Write in sigma notation [15]

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} = \sum_{j=1}^{5} (-1)^{j-1} \frac{1}{j^2}.$$

TURN OVER FOR QUESTIONS 4 AND 5

4. A ship moves along the x-axis in the positive direction at $4km/hr$, [30]

and another ship moves along the y-axis in the positive direction at $3km/hr$. At time 0 both ships start at the origin. How fast is the distance d between them increasing after $1hr$?

Solution: Denote the sides of our right triangle by x (the horizontal side), y (the vertical side), and d (the hypothenus). By the **Pythagorean Theorem**, we have the **basic relation**:

$$x^2 + y^2 = d^2.$$

Let us differentiate this basic relation with respect to time t using the Chain Rule, remembering that x, y, and d are functions of t:

$$2xx' + 2yy' = 2dd', \qquad xx' + yy' = dd', \qquad d' = \frac{xx' + yy'}{d}.$$

We want to find d' at time $t = 1$. At this time, $x = 4 \cdot 1 = 4$ and $y = 3 \cdot 1 = 3$, so $d = \sqrt{4^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25} = 5$. We also know that $x' = 4$ and $y' = 3$.

Now evaluate at $t = 1$:

$$d'(1) = \left.\frac{xx' + yy'}{d}\right|_{t=1} = \frac{4 \cdot 4 + 3 \cdot 3}{5} = \frac{16 + 9}{5} = \frac{25}{5} = 5.$$

Answer: 5 km/hr.

5. Evaluate [20]

$$\sum_{k=0}^{n}(2k+1) = 2\left(\sum_{k=0}^{n} k\right) + \left(\sum_{k=0}^{n} 1\right)$$

$$= 2(1 + 2 + 3 + \ldots + (n-1) + n) + \underbrace{(1 + 1 + \ldots + 1 + 1)}_{\leftarrow\ (n+1) \text{ times } \rightarrow}$$

$$= 2\frac{n(n+1)}{2} + (n+1) = n(n+1) + (n+1) = (n+1)(n+1) = (n+1)^2.$$

END OF PAPER