MATH 251

WORKSHEET #2 - SOLUTIONS

1. For $f(x) = \frac{x+3}{5x+x^2+6}$ find

$$\text{a: } \lim_{x\to -2^-} f(x), \qquad \text{b: } \lim_{x\to -\infty} f(x), \qquad \text{c: } \lim_{x\to -3} f(x),$$

find the horizontal and vertical asymptotes, and sketch the graph of $y = f(x)$.

Solution: First establish the types of limits:

$$\text{a: } "\frac{1}{0}", \qquad \text{b: } "\frac{-\infty}{\infty}", \qquad \text{c: } "\frac{0}{0}".$$

We see that the first limit DOES NOT EXIST, but we still must find out whether it is ∞ or $-\infty$ or neither. Since all limits have indefinite types, we must simplify $f(x)$ for $x \neq -2, -3$. While calculating the types of the first and the third limit, we found out that the denominator has roots at $x = -2$ and $x = -3$. Therefore $(x+2)$ and $(x+3)$ are factors of the denominator, so $5x+x^2+6 = (x+3)(x+2)$. Cancel the common factor: $f(x) = \frac{x+3}{(x+3)(x+2)} = \frac{1}{x+2}$. Now

for a: $$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} \frac{1}{x+2} = "\frac{1}{0^-}" = -\infty,$$

and the limit DOES NOT EXIST.

for b: $$\lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} \frac{1}{x+2} = "\frac{1}{-\infty}" = 0.$$

for c: $$\lim_{x\to -3} f(x) = \lim_{x\to -3} \frac{1}{x+2} = \frac{1}{-1} = -1.$$

Since $f(x) = \frac{1}{x+2}$ for $x \neq -2, -3$, the graph of $y = f(x)$ is a hyperbola with one missing point $(-3, -1)$ which is obtained from the hyperbola $y = \frac{1}{x}$ by the shift of two units to the left. So, the graph has a vertical asymptote $x = -2$ and a horizontal asymptote $y = 0$.

2. For $g(x) = \frac{2-\sqrt{3x^2+1}}{x-1}$ find

$$\text{a: } \lim_{x\to 1} g(x), \qquad \text{b: } \lim_{x\to -\infty} g(x), \qquad \text{c: } \lim_{x\to \infty} g(x).$$

Solution: First establish the types of limits:

$$\text{a: } ”\frac{0}{0}”, \qquad \text{b: } ”\frac{-\infty}{-\infty}”, \qquad \text{c: } ”\frac{-\infty}{\infty}”.$$

Since all limits have indefinite types, we must simplify $g(x)$ for $x \neq 1$. Multiply the numerator and denominator by the conjugate of the numerator:

$$g(x) = \frac{2-\sqrt{3x^2+1}}{x-1} = \frac{(2-\sqrt{3x^2+1})(2+\sqrt{3x^2+1})}{(x-1)(2+\sqrt{3x^2+1})} = \frac{4-(3x^2+1)}{(x-1)(2+\sqrt{3x^2+1})}$$

$$= \frac{3-3x^2}{(x-1)(2+\sqrt{3x^2+1})} = \frac{3(1-x^2)}{(x-1)(2+\sqrt{3x^2+1})} = \frac{3(1-x)(1+x)}{(x-1)(2+\sqrt{3x^2+1})}.$$

Cancel the common factor $(x-1)$:

$$g(x) = \frac{-3(1+x)}{2+\sqrt{3x^2+1}}.$$

Now

for a: $$\lim_{x\to 1} g(x) = \lim_{x\to 1} \frac{-3(1+x)}{2+\sqrt{3x^2+1}} = \frac{-3\cdot 2}{2+2} = -\frac{3}{2}.$$

for b: $$\lim_{x\to -\infty} g(x) = \lim_{x\to -\infty} \frac{-3(1+x)}{2+\sqrt{3x^2+1}} = ”\frac{\infty}{\infty}”,$$

so the limit is indefinite. Therefore we must simplify $g(x)$ by dividing the denominator and numerator by the highest power of x in the denominator:

$$g(x) = \frac{-3(\frac{1}{x}+1)}{\frac{2}{x}+\frac{\sqrt{3x^2+1}}{x}}.$$

Since $x \to -\infty$, x is negative, so we have $x = -\sqrt{x^2}$ (with the minus sign), so

$$g(x) = \frac{-3(\frac{1}{x}+1)}{\frac{2}{x}-\sqrt{\frac{3x^2+1}{x^2}}} = \frac{-3(\frac{1}{x}+1)}{\frac{2}{x}-\sqrt{3+\frac{1}{x^2}}}.$$

Now let us substitute $x = -\infty$ using the rule ”$\frac{1}{\infty} = 0$”:

$$\lim_{x\to -\infty} g(x) = \frac{-3(0+1)}{0-\sqrt{3+0}} = \frac{3}{\sqrt{3}} = \frac{\sqrt{3}\sqrt{3}}{\sqrt{3}} = \sqrt{3}.$$

for c:
$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{-3(1+x)}{2+\sqrt{3x^2+1}} = \text{"}\frac{-\infty}{\infty}\text{"},$$

so the limit is indefinite similarly to (b). We simplify $g(x)$ as in (b), so

$$g(x) = \frac{-3(\frac{1}{x}+1)}{\frac{2}{x}+\frac{\sqrt{3x^2+1}}{x}}.$$

Since $x \to \infty$, x is positive, so we have $x = \sqrt{x^2}$ (with the plus sign), so

$$g(x) = \frac{-3(\frac{1}{x}+1)}{\frac{2}{x}+\sqrt{\frac{3x^2+1}{x^2}}} = \frac{-3(\frac{1}{x}+1)}{\frac{2}{x}+\sqrt{3+\frac{1}{x^2}}}.$$

Now let us substitute $x = \infty$ using the rule "$\frac{1}{\infty} = 0$":

$$\lim_{x\to\infty} g(x) = \frac{-3(0+1)}{0+\sqrt{3+0}} = \frac{-3}{\sqrt{3}} = \frac{-\sqrt{3}\sqrt{3}}{\sqrt{3}} = -\sqrt{3}.$$

Since $g(x) = \frac{2-\sqrt{3x^2+1}}{x-1}$, the only vertical asymptote of the graph of $y = g(x)$ can be $x = 1$. But the limit in (a) is finite, so $x = 1$ is not a vertical asymptote. So the graph has no vertical asymptotes. By (b) and (c) the horizontal asymptotes are $y = \sqrt{3}$ and $y = -\sqrt{3}$.

3. Find the limits and state your answers as ∞ or $-\infty$ where appropriate.

$$\text{a}: \lim_{x\to 3} \frac{x^2-9}{x^2-x-6}, \qquad \text{b}: \lim_{x\to -\infty}(x^3-3x^5), \qquad \text{c}: \lim_{x\to 1^-} \frac{x^2-2}{x^3+2x-3},$$

$$\text{d}: \lim_{x\to -3} \frac{x^3+27}{x^3-6x+9}, \qquad \text{e}: \lim_{x\to 0} \frac{1-|1-2x|}{x^2-2x}, \qquad \text{f}: \lim_{x\to\infty}(\sqrt{9x^2+2x}-3x).$$

Solution: First establish the types of limits:

$$\text{a: "}\frac{0}{0}\text{"}, \qquad \text{b: "}-\infty+\infty\text{"}, \qquad \text{c: "}\frac{-1}{0}\text{"},$$

$$\text{d: "}\frac{0}{0}\text{"}, \qquad \text{e: "}\frac{0}{0}\text{"}, \qquad \text{f: "}\infty-\infty\text{"}.$$

We see that the limit in (c) DOES NOT EXIST, but we still must find out whether it is ∞ or $-\infty$ or neither. Since all limits have indefinite types, we must rewrite the function in a form having a definite limit:

$$\text{a}: \lim_{x\to 3} \frac{x^2-9}{x^2-x-6} = \text{"}\frac{0}{0}\text{"} = \lim_{x\to 3} \frac{(x-3)(x+3)}{(x-3)(x+2)} = \lim_{x\to 3} \frac{x+3}{x+2} = \frac{6}{5}.$$

$$\text{b}: \lim_{x\to-\infty}(x^3-3x^5) = \text{"}-\infty+\infty\text{"} = \lim_{x\to-\infty} x^5(\frac{1}{x^2}-3) = \text{"}(-\infty)\cdot(0-3)\text{"} = \text{"}\infty\cdot 3\text{"} = \infty$$

and the limit DOES NOT EXIST.

$$\text{c}: \lim_{x\to 1^-} \frac{x^2-2}{x^3+2x-3} = \text{"}\frac{-1}{0}\text{"} = \lim_{x\to 1^-} \frac{x^2-2}{(x-1)(x^2+x+3)} = \text{"}\frac{-1}{(0^-)\cdot 5}\text{"} = \text{"}\frac{-1}{0^-}\text{"} = \infty$$

and the limit DOES NOT EXIST.

$$\text{d}: \lim_{x\to-3} \frac{x^3+27}{x^3-6x+9} = \text{"}\frac{0}{0}\text{"} = \lim_{x\to-3} \frac{(x+3)(x^2-3x+9)}{(x+3)(x^2-3x+3)} = \lim_{x\to-3} \frac{x^2-3x+9}{x^2-3x+3} = \frac{9}{7}.$$

$$\text{e}: \lim_{x\to 0} \frac{1-|1-2x|}{x^2-2x} = \text{"}\frac{0}{0}\text{"} = \lim_{x\to 0} \frac{1-(1-2x)}{x^2-2x} = \lim_{x\to 0} \frac{2x}{x^2-2x} = \lim_{x\to 0} \frac{2}{x-2} = \frac{2}{0-2} = -1.$$

$$\text{f}: \lim_{x\to\infty}(\sqrt{9x^2+2x}-3x) = \text{"}\infty-\infty\text{"} = \lim_{x\to\infty} \frac{(\sqrt{9x^2+2x}-3x)(\sqrt{9x^2+2x}+3x)}{\sqrt{9x^2+2x}+3x}$$

$$= \lim_{x\to\infty} \frac{9x^2+2x-9x^2}{\sqrt{9x^2+2x}+3x} = \lim_{x\to\infty} \frac{2x}{\sqrt{9x^2+2x}+3x} = \text{"}\frac{\infty}{\infty}\text{"}$$

$$= \lim_{x\to\infty} \frac{2}{\frac{\sqrt{9x^2+2x}}{x}+3} = \lim_{x\to\infty} \frac{2}{\sqrt{9+\frac{2}{x}}+3} = \frac{2}{\sqrt{9+0}+3} = \frac{2}{3+3} = \frac{1}{3}.$$

4. a: Find $\lim_{x\to-\infty} \frac{3x+\sin(x+\frac{2}{x})}{x+3}$.

b: Suppose that $6x \le f(x) \le x^2+9$ for all x sufficiently close to 3, find $\lim_{x\to 3} f(x)$.

c: Suppose that $g(x) \ge \frac{\sqrt{x^2-3}}{x^2-x-2}$ for all $x > 2$, find $\lim_{x\to 2^+} g(x)$.

Solution: We use the Squeeze Theorem for this group of limits.
for a:

$$\lim_{x\to-\infty} \frac{3x+\sin(x+\frac{2}{x})}{x+3} = \text{"}\frac{-\infty}{-\infty}\text{"} = \lim_{x\to-\infty} \frac{3+\frac{\sin(x+\frac{2}{x})}{x}}{1+\frac{3}{x}} = \frac{3+0}{1+0} = 3.$$

Here we calculated the limit of $\frac{\sin(x+\frac{2}{x})}{x}$ as $x\to-\infty$ by using the Squeeze Theorem: for negative x we have

$$\frac{1}{x} \le \frac{\sin(x+\frac{2}{x})}{x} \le \frac{-1}{x}$$

and $\lim_{x\to-\infty}\frac{1}{x}=0$, $\lim_{x\to-\infty}\frac{-1}{x}=0$. Therefore $\lim_{x\to-\infty}\frac{\sin(x+\frac{2}{x})}{x}=0$ by Squeeze Theorem.
for b: The function f is squeezed between the functions $6x$ and x^2+9. Find the limits of the squeezing functions as $x\to 3$:

$$\lim_{x\to 3}(6x)=18, \qquad \lim_{x\to 3}(x^2+9)=18.$$

Since the limit is the same, the function f is forced to have the same limit. Therefore $\lim_{x\to 3}f(x)=18$.
for c: First find the limit of the "squeezing function":

$$\lim_{x\to 2^+}\frac{\sqrt{x^2-3}}{x^2-x-2}=\text{"}\frac{1}{0}\text{"}=\lim_{x\to 2^+}\frac{\sqrt{x^2-3}}{(x-2)(x+1)}=\text{"}\frac{1}{(0^+)\cdot 3}\text{"}=\infty$$

So, $\frac{\sqrt{x^2-3}}{x^2-x-2}$ becomes very large as $x\to 2^+$, and therefore $g(x)$ is forced to become very large too. Therefore $\lim_{x\to 2^+}g(x)=\infty$ and the limit DOES NOT EXIST.

5. a: The Intermediate-Value Theorem allows us to conclude that the equation $\sqrt[3]{x+2}=3-x$ has a root in the interval

(a) $[-1,0]$, (b) $[0,1]$, (c) $[1,2]$, (d) $[2,3]$, (e) None of these.

b: Consider the equation $x^3+2x-22=0$. Let r denote a root of the equation. Find an estimate for r with Error $<\frac{1}{4}$. **Hint:** Use the bisection method.

Solution:
for a: Rewrite our equation in the form $x+2=(3-x)^3$, so $(x-3)^3+x+2=0$. So, we want to find a root of $f(x)=(x-3)^3+x+2$. Since f is a polynomial, it is continuous. Let us test the values of $f(x)$ at the points $x=-1,0,1,2,3$: we get $f(-1)=-4^3-1+2=-64-1+2=-63$, $f(0)=-3^3+2=-27+2=-25$, $f(1)=-2^3+1+2=-8+1+2=-5$, $f(2)=-1+2+2=3$, $f(3)=0+3+2=5$.

So, the values of f have different signs at the endpoints of the interval $[1,2]$. The Intermediate-Value Theorem allows us to conclude that the equation $f(x)=0$ has a root in this interval.
for b: Consider the function $g(x)=x^3+2x-22$. Again, since g is a polynomial, it is continuous. We want to estimate the location of the root r of g. Surely r must be positive.

Let us check the signs of $g(x)$ for some integer values of x. Since $g(0) = -22$ is negative, and $g(x)$ becomes larger and larger as x increases, it has to become positive at some integer point $x > 0$:

$$g(0) = -22, \quad g(1) = 1+2-22 = -19, \quad g(2) = 8+4-22 = -10, \quad g(3) = 27+6-22 = 11.$$

So, the values of g have different signs at the endpoints of the interval $[2, 3]$. The Intermediate-Value Theorem allows us to conclude that the equation $g(x) = 0$ has a root r in the interval $(2, 3)$.

Now we will specify the location of r more accurately by using the bisection method: find the sign of $g(x)$ at the center of the interval $[2, 3]$:

$$g(2.5) = 2.5^3 + 2 \cdot 2.5 - 22 = \frac{125}{8} + 5 - 22 = \frac{125 + 40 - 176}{8} = -\frac{11}{8}.$$

Since $g(2.5)$ is negative and $g(3)$ is positive, the Intermediate-Value Theorem allows us to conclude that the equation $g(x) = 0$ has a root r in the interval $(2.5, 3)$. Let us take the centre of this interval 2.75 as an estimate for r. So $r \approx 2.75$ and the Error of our estimate is $|r - 2.75|$. Since $2.5 < r < 3$, we have $2.75 - 0.25 < r < 2.75 + 0.25$. Therefore $|r - 2.75| < 0.25$, so $|r - 2.75| < \frac{1}{4}$, so our estimate has Error $< \frac{1}{4}$.

Answer: $r \approx 2.75$ with Error $< \frac{1}{4}$.

6. a: Let ε be a small positive real number. How close to 4 must we hold x to be sure that $\sqrt{x+5}$ lies within ε units of 3?

b: Let ε be a small positive real number, and let $x > 0$. How large must we hold x to be sure that $\frac{x}{x+2}$ lies within ε units of 1?

Solution:

for a: We want to hold x sufficiently close to 4 to be sure that $|\sqrt{x+5} - 3| \leq \varepsilon$. Let us solve this inequality for x:

$$3 - \varepsilon \leq \sqrt{x+5} \leq 3 + \varepsilon, \qquad (3-\varepsilon)^2 \leq x + 5 \leq (3+\varepsilon)^2,$$

$$(3-\varepsilon)^2 - 5 \leq x \leq (3+\varepsilon)^2 - 5, \qquad 4 - 6\varepsilon + \varepsilon^2 \leq x \leq 4 + 6\varepsilon + \varepsilon^2,$$

so $x \in [4 - 6\varepsilon + \varepsilon^2,\ 4 + 6\varepsilon + \varepsilon^2]$. If we will now hold x within $6\varepsilon - \varepsilon^2$ units of 4 then it will surely lie in the interval $[4 - 6\varepsilon + \varepsilon^2,\ 4 + 6\varepsilon + \varepsilon^2]$.

Answer: We must hold x within $6\varepsilon - \varepsilon^2$ units of 4.

for b: We want to hold x sufficiently large to be sure that $|\frac{x}{x+2} - 1| \leq \varepsilon$. Let us solve this inequality for x: $\quad |\frac{-2}{x+2}| \leq \varepsilon, \quad \frac{2}{x+2} \leq \varepsilon, \quad \frac{x+2}{2} \geq \frac{1}{\varepsilon}, \quad x + 2 \geq \frac{2}{\varepsilon}$, $x \geq \frac{2}{\varepsilon} - 2$.

Answer: We must hold $x \geq \frac{2}{\varepsilon} - 2$.

7. The function $\operatorname{sgn} x = \frac{x}{|x|}$ is neither continuous nor discontinuous at $x = 0$. How is this possible?

Solution: Recall that a function $f(x)$ is called continuous at $x = 0$ if $f(0) = \lim_{x\to 0} f(x)$. So the point $x = 0$ must lie in the domain of $f(x)$ if we want to speak about whether $f(x)$ is continuous or discontinuous at $x = 0$. But the function $\operatorname{sgn} x$ is not defined at $x = 0$. So it can be neither continuous nor discontinuous at the point $x = 0$, since this point does not lie in the domain of the function.