MATH 251

WORKSHEET #4 - SOLUTIONS

1. $f(x) = \frac{1}{x} + \sin x$, $f'(x) = -\frac{1}{x^2} + \cos x$, $f''(x) = (f'(x))' = \frac{2}{x^3} - \sin x$.

2. a: First check that the point $(1, 0)$ satisfies the given equation: for $x = 1, y = 0$ the left hand side equals $\tan(0) + \sin 0 = 0 + 0 = 0$ and the right hand side equals $0 + 2 - 2 = 0$, so the point $(1, 0)$ belongs to the graph. Let us differentiate both sides of $\tan(xy) + \sin y = x^2y + 2x - 2$ with respect to x using the chain rule, remembering that y is a function of x:

$$\sec^2(xy) \cdot (y + xy') + \cos y \cdot y' = 2xy + x^2y' + 2.$$

In order to solve this equation for y', we bring all terms containing y' to the left, and bring all terms which do not contain y' to the right:

$$(x \sec^2(xy) + \cos y - x^2)y' = -y \sec^2(xy) + 2xy + 2, \qquad y' = \frac{x \sec^2(xy) + \cos y - x^2}{-y \sec^2(xy) + 2xy + 2}.$$

Substituting $x = 1, y = 0$, we obtain the slope of the tangent line at the point $(1, 0)$:

$$m = y'(1) = \left.\frac{x \sec^2(xy) + \cos y - x^2}{-y \sec^2(xy) + 2xy + 2}\right|_{\substack{x=1\\y=0}} = \frac{\sec 0 + \cos 0 - 1}{\sec 0 + 0 + 2} = \frac{1+1-1}{1+0+2} = \frac{1}{3}.$$

So, the tangent line at the point $(1, 0)$ has slope $m = \frac{1}{3}$. Therefore the equation of this tangent line is $y = m(x - 1) = \frac{x-1}{3}$.

Answer: $y = \frac{x-1}{3}$.

b: First check that the point $(2, -2)$ satisfies the given equation: for $x = 2, y = -2$ the left hand side equals $2 \cdot 2^2 - 2^2(-2) = 8 + 8 = 16$, so the point $(2, -2)$ belongs to the graph. Let us differentiate both sides of $xy^2 - x^2y = 16$ with respect to x using the chain rule, remembering that y is a function of x:

$$y^2 + x \cdot 2yy' - (2xy + x^2y') = 0.$$

In order to solve this equation for y', we bring all terms containing y' to the left, and bring all terms which do not contain y' to the right:

$$(2xy - x^2)y' = -y^2 + 2xy, \qquad y' = \frac{-y^2 + 2xy}{2xy - x^2}.$$

Substituting $x = 2, y = -2$, we obtain the slope of the tangent line at the point $(2, -2)$:

$$m = y'(2) = \left.\frac{-y^2 + 2xy}{2xy - x^2}\right|_{\substack{x=2\\y=-2}} = \frac{-2^2 - 2 \cdot 2 \cdot 2}{-2 \cdot 2 \cdot 2 - 2^2} = \frac{-4-8}{-8-4} = 1.$$

So, the tangent line at the point $(2, -2)$ has slope $m = 1$. Therefore the equation of this tangent line is $y = m(x - 2) - 2 = x - 4$.

Answer: $y = x - 4$.

3. b: Since we want to find both y' and y'', let us differentiate both sides of $x^3 + y^3 = 27$ with respect to x two times, remembering that y is a function of x:

$$3x^2 + 3y^2y' = 0, \qquad 6x + 3(2yy'y' + y^2y'') = 0.$$

In order to solve the first equation for y', we bring all terms containing y' to the left, and bring all terms which do not contain y' to the right:

$$3y^2y' = -3x^2, \qquad y' = -\frac{x^2}{y^2}.$$

In order to solve the second equation for y'', we bring all terms containing y'' to the left, and bring all terms which do not contain y'' to the right:

$$3y^2y'' = -6x - 6y(y')^2, \qquad y'' = -2\frac{x + y(y')^2}{y^2}.$$

Substitute the expression for y' from above:

$$y'' = -2\frac{x + y(\frac{x^2}{y^2})^2}{y^2} = -2\frac{x + \frac{x^4}{y^3}}{y^2} = -2\frac{xy^3 + x^4}{y^5} = -2\frac{x}{y^5}(x^3 + y^3)$$

Since $x^3 + y^3 = 27$, we have

$$y'' = -54\frac{x}{y^5}.$$

Answer: $y' = -\frac{x^2}{y^2}$, $y'' = -54\frac{x}{y^5}$.

a: Since we want to find both y' and y'', let us differentiate both sides of $x^2y + xy^2 = 6$ with respect to x two times, remembering that y is a function of x:

$$2xy + x^2y' + y^2 + x\cdot 2yy' = 0, \qquad 2(y + xy') + 2xy' + x^2y'' + 2yy' + 2(yy' + xy'y' + xyy'') = 0.$$

In order to solve the first equation for y', we bring all terms containing y' to the left, and bring all terms which do not contain y' to the right:

$$(x^2 + 2xy)y' = -2xy - y^2, \qquad y' = \frac{-2xy - y^2}{x^2 + 2xy}, \qquad y' = -\frac{2xy + y^2}{x^2 + 2xy}.$$

In order to solve the second equation for y'', we bring all terms containing y'' to the left, and bring all terms which do not contain y'' to the right:

$$(x^2 + 2xy)y'' = -(2(y + xy') + 2xy' + 2yy' + 2(yy' + xy'y')),$$

$$y'' = -\frac{2(y + xy') + 2xy' + 2yy' + 2(yy' + xy'y')}{x^2 + 2xy}.$$

Simplify and substitute the expression for y' from above:

$$y'' = -\frac{2y + 4(x+y)y' + 2x(y')^2}{x^2 + 2xy} = -\frac{2y - 4(x+y)\frac{2xy+y^2}{2xy+x^2} + 2x(\frac{2xy+y^2}{2xy+x^2})^2}{x^2 + 2xy}.$$

Answer: $y' = -\frac{2xy+y^2}{x^2+2xy}$, $y'' = -\frac{2y-4(x+y)\frac{2xy+y^2}{2xy+x^2}+2x(\frac{2xy+y^2}{2xy+x^2})^2}{x^2+2xy}$.

4. (a) $\int x^{12}dx = \frac{x^{13}}{13} + C$

(b) $\int \frac{x^2+4}{2x}dx = \int(\frac{x}{2} + \frac{2}{x})dx = \frac{x^2}{4} + 2\ln|x| + C$

(c) $\int(x^2 + \sec^2 x + e^x)dx = \frac{x^3}{3} + \tan x + e^x + C$

(d) $\int 2x\sin(x^2)dx = -\cos(x^2) + C$

(e) $\int(3 + x^{-4} - x^{-1} + e^{2x})dx = 3x - \frac{x^{-3}}{3} - \ln|x| + \frac{1}{2}e^{2x} + C$

5. First we solve the differential equation. Since $y'(x) = x^{1/3}$, the function $y(x)$ is the anti-derivative of $x^{1/3}$, so

$$y(x) = \int x^{1/3}dx = \frac{3}{4}x^{4/3} + C$$

where C is an arbitrary constant. Now we find C using the initial condition: since $y(0) = C$, we have $C = 5$.

Answer: $y(x) = \frac{3}{4}x^{4/3} + 5$.

6. (a) $v(t) = s'(t) = (te^{-t})' = e^{-t} + t(-e^{-t}) = (1-t)e^{-t}$,
$a(t) = v'(t) = ((1-t)e^{-t})' = -e^{-t} + (1-t)(-e^{-t}) = (-2+t)e^{-t}$.

(b) Speeding up means $a(t) > 0$, so $-2+t > 0$, $t > 2$, $t \in (2, \infty)$. Slowing down means $a(t) < 0$, so $-2+t < 0$, $t < 2$, $t \in [0, 2)$. Here $t \geq 0$, since the position of the body is given for $t \geq 0$ only.

7. **(a)** The function f is defined for any $x \neq -1$, so its domain is $D_f = (-\infty, -1) \cup (-1, \infty)$. Since $f(x) = \frac{x}{1+x} = \frac{1+x-1}{1+x} = 1 - \frac{1}{1+x}$, the graph of f is a hyperbola with the vertical asymptote $x = -1$ and the horizontal asymptote $y = 1$. So, any number $y \neq 1$ is an output value of $y = f(x)$. Thus the range of f is $R_f = (-\infty, 1) \cup (1, \infty)$. Now

$$R_g = D_f = (-\infty, -1) \cup (-1, \infty), \qquad D_g = R_f = (-\infty, 1) \cup (1, \infty).$$

Denote $y = f(x) = \frac{x}{1+x}$. To find the inverse function g, we must rewrite this equation in the form $x = g(y)$. So, we must solve the equation $y = \frac{x}{1+x}$ for x:

$$y = \frac{x}{1+x}, \qquad y(1+x) = x, \qquad (y-1)x = -y, \qquad x = -\frac{y}{y-1}.$$

Therefore, $g(y) = -\frac{y}{y-1}$.

(b) Since $g'(y) = \frac{1}{f'(x)}\Big|_{x=g(y)}$, we have

$$g'(2) = \frac{1}{f'(x)}\Big|_{x=g(2)}.$$

In order to find $g(2)$, observe that the equation $y = f(x)$ is equivalent to the equation $x = g(y)$, thus $x = g(2)$ is equivalent to $2 = f(x)$. Let us solve this equaition for x:

$$\frac{4x^3}{x^2+1} = 2, \qquad 4x^3 = 2x^2 + 2, \qquad 2x^3 - x^2 - 1 = 0.$$

Observe that $x = 1$ is a root of this equation, so $(x-1)$ is a factor of the left hand side. To find another factor, we divide $x - 1$ into $2x^3 - x^2 - 1$ using long division of polynomials, this gives $2x^3 - x^2 - 1 = (x-1)(2x^2 + x + 1)$. Since the discriminant of the second factor is $1 - 4 \cdot 2 = -7 < 0$, the second factor is always positive. Therefore $x = 1$ is the only root, so $g(2) = 1$. Now

$$g'(2) = \frac{1}{f'(x)}\Big|_{x=1} = \frac{1}{f'(1)}.$$

Let us compute $f'(1)$:

$$f'(x) = \left(\frac{4x^3}{x^2+1}\right)' = \frac{4 \cdot 3x^2(x^2+1) - 2x(4x^3)}{(x^2+1)^2} = \frac{4x^4 + 12x^2}{(x^2+1)^2},$$

so $f'(1) = \frac{4+12}{2^2} = \frac{16}{4} = 4$. Thus $g'(2) = \frac{1}{4}$.

Answer: $g'(2) = \frac{1}{4}$.

(c) By the formula for the derivative of the inverse function, $g'(y) = \frac{1}{f'(x)}\Big|_{x=g(y)}$. Let us compute $f'(x)$:

$$f'(x) = \sqrt{3+x^2} + x\frac{1}{2\sqrt{3+x^2}} = \frac{2(3+x^2)+x}{2\sqrt{3+x^2}}.$$

Thus

$$g'(y) = \frac{2\sqrt{3+x^2}}{2(3+x^2)+x}\Big|_{x=g(y)}.$$

One should now replace x by $g(y)$, the algebra is complicated.

8. (a) $f(x) = \frac{e^x}{x}$. This function is defined for any $x \neq 0$, since e^x is defined for any $x \in \mathbb{R}$. Thus $D_f = (-\infty, 0) \cup (0, \infty)$.

(b) $g(x) = \sqrt{9 - x^2}$. This function is defined for all x such that $9 - x^2 \geq 0$. Let us solve this inequality: $(3-x)(3+x) \geq 0$, so the splitting points are $x = -3$ and $x = 3$, and testing shows that $x \in (-\infty, -3] \cup [3, \infty)$. Thus $D_g = (-\infty, -3] \cup [3, \infty)$.

(c) $h(x) = \ln(\ln x)$. The function $\ln x$ is defined for any $x > 0$. Therefore the argument of each "ln" in the expression for g should be positive. The argument of the first "ln" is x, and the argument of the second "ln" is $\ln x$. Thus the function $h(x)$ is defined for all x satisfying the following system of inequalities:

$$x > 0, \qquad \ln x > 0.$$

Actually this system is equivalent just to the inequality $\ln x > 0$, since $\ln x$ is defined for $x > 0$ only. So we must solve the inequality $\ln x > 0$. The splitting point is $x = 1$, since $\ln 1 = 0$. Observe that the function $\ln x$ is increasing, since $(\ln x)' = \frac{1}{x} > 0$ for $x > 0$. Therefore the inequality $\ln x > 0$ is equivalent to $x \in (1, \infty)$. Thus $D_h = (1, \infty)$.

9. (a) $y = xe^x - x, \quad y' = e^x + xe^x - 1$

(b) $y = \ln|3x - 2|, \quad y' = \dfrac{1}{3x-2} \cdot 3 = \dfrac{3}{3x-2}$

(c) $y = \ln \ln x, \quad y' = \dfrac{1}{\ln x} \cdot \dfrac{1}{x} = \dfrac{1}{x \ln x}$

(d) $y = x^2 \ln(e^x + 1), \quad y' = 2x \ln(e^x + 1) + x^2 \dfrac{1}{e^x + 1} e^x = 2x \ln(e^x + 1) + \dfrac{x^2 e^x}{e^x + 1}$

10. $5e^{-\ln 5} - \ln(e^{-5}) = 5(e^{\ln 5})^{-1} - \ln(e^{-5})$

Since the functions $\ln x$ and e^x are the inverses of each other, $\ln(e^x) = x$ and $e^{\ln x} = x$. So,

$$= 5 \cdot 5^{-1} - (-5) = \frac{5}{5} + 5 = 1 + 5 = 6.$$

Answer: 6.