MATH 251

WORKSHEET #5 – SOLUTIONS

1. Simplify:

(a)

$$\log_{10} 1000 + \log_{10} 0.01 = 3 + (-2) = 1 \quad OR \quad = \log_{10}(1000 \cdot 0.01) = \log_{10} 10 = 1,$$

(b)
$$(x^{-3})^{-2} = x^{(-3)(-2)} = x^6,$$

(c)
$$2^{\log_4 8} = 2^{\frac{\log_2 8}{\log_2 4}} = 2^{\frac{3}{2}} = (2^{\frac{1}{2}})^3 = (\sqrt{2})^3 = 2\sqrt{2},$$

(d)
$$\log_6 9 + \log_6 4 = \log_6(9 \cdot 4) = \log_6(36) = \log_6(6^2) = 2,$$

(e)
$$\tan(\arctan 200) = 200,$$

(f)
$$\arccos(\sin(-0.2)) = \frac{\pi}{2} - \arcsin(\sin(-0.2)) = \frac{\pi}{2} - (-0.2) = \frac{\pi}{2} + 0.2$$

(g)
$$\tan(\arcsin x) = \tan y$$

where $y = \arcsin x$, so $\sin y = x$. From the "Right Triangle method" (draw the picture!) we find $\tan y = \frac{x}{\sqrt{1-x^2}}$, so

$$\tan(\arcsin x) = \tan y = \frac{x}{\sqrt{1-x^2}}.$$

3. Find the following limits:

(a)
$$\lim_{x\to 1^-} \log_x 2 = \lim_{x\to 1^-} \frac{\ln 2}{\ln x} = \text{“}\frac{\ln 2}{\ln(1^-)}\text{”} = \text{“}\frac{\ln 2}{0^-}\text{”} = \text{“}\frac{1}{0^-}\text{”} = -\infty, \qquad \textbf{DNE}.$$

Here we used that $\ln x$ is increasing (since $(\ln x)' = \frac{1}{x} > 0$), so $\ln 2 > \ln 1 = 0$, and $\ln(1^-) = (\ln 1)^- = 0^-$.

(b)
$$\lim_{x\to\infty} x^3 e^{-x} = \text{“}\infty \cdot 0\text{”} = 0,$$

since the exponential growth is much faster than the polynomial growth. (Remember that, in a battle between a polynomial and an exponential, the exponential always wins!)

2. In questions (a) and (b), we use logarithmic differentiation:

(a) $$y = (\cos x)^x, \qquad \ln y = x\ln(\cos x),$$

now differentiate both sides of this equality with respect to x, remembering that y is a function of x:

$$\frac{y'}{y} = \ln(\cos x) + x\frac{1}{\cos x}\cdot(-\sin x), \quad \frac{y'}{y} = \ln(\cos x) - x\cot x,$$
$$y' = y(\ln(\cos x) - x\cot x) = (\cos x)^x(\ln(\cos x) - x\cot x).$$

(b) $$y = x^{\ln x}, \qquad \ln y = (\ln x)(\ln x) = (\ln x)^2,$$

now differentiate both sides of this equality with respect to x, remembering that y is a function of x:

$$\frac{y'}{y} = 2(\ln x)\cdot\frac{1}{x}, \quad \frac{y'}{y} = \frac{2\ln x}{x}, \quad y' = y\frac{2\ln x}{x} = x^{\ln x}\frac{2\ln x}{x} = 2x^{-1+\ln x}\ln x.$$

(c) $$y = x\arctan x, \qquad y' = \arctan x + x(\arctan x)' = \arctan x + \frac{x}{1+x^2},$$

where we used the formula $(\arctan x)' = \frac{1}{1+x^2}$. Let us show how to get this formula using the formula for the derivative of the inverse function, and the fact that $(\tan u)' = \sec^2 u$:

$$u = f(x) = \arctan x, \quad x = g(u) = \tan u, \quad f'(x) = \frac{1}{g'(f(x))} = \left.\frac{1}{g'(u)}\right|_{u=f(x)},$$

so

$$(\arctan x)' = \left.\frac{1}{(\tan u)'}\right|_{u=\arctan x} = \left.\frac{1}{\sec^2 u}\right|_{u=\arctan x} = \cos^2 u\big|_{u=\arctan x} = \cos^2(\arctan x)$$

To find $\cos(\arctan x)$, denote $u = \arctan x$, so $\tan u = x$, and using the "Right Triangle method" (draw the picture!) we get $\cos u = \frac{1}{\sqrt{1+x^2}}$, so $\cos^2 u = \frac{1}{1+x^2}$, thus we get

$$= \cos^2 u = \frac{1}{1+x^2}.$$

(d) $$y = x^2\arcsin x, \qquad y' = 2x\arcsin x + x^2(\arcsin x)' = 2x\arcsin x + \frac{x^2}{\sqrt{1-x^2}},$$

where we used the formula $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$. Let us show how to get this formula, using the formula for the derivative of the inverse function (see (c)), and the fact that $(\sin u)' = \cos u$: so

$$(\arcsin x)' = \left.\frac{1}{(\sin u)'}\right|_{u=\arcsin x} = \left.\frac{1}{\cos u}\right|_{u=\arcsin x} = \frac{1}{\cos(\arcsin x)}$$

To find $\cos(\arcsin x)$, denote $u = \arcsin x$, so $\sin u = x$, and using the "Right Triangle method" (draw the picture!) we get $\cos u = \sqrt{1-x^2}$, thus we get

$$= \frac{1}{\cos u} = \frac{1}{\sqrt{1-x^2}}.$$

4. The point P is moving along the circle $x^2 + y^2 = 1m^2$ clockwise with angular velocity $3\,rad/s$. How fast is the distance of P from the point $Q = (0, 2)m$ changing when $P = (1, 0)m$?

Solution:

Step 1 (reducing the question to a basic relation $F(x, y) = 0$ between two appropriate functions x, y of time t).
The position of the particle P at time t is given by its coordinates $P(x(t), y(t))$ which are related to each other by the relation $x^2 + y^2 = 1$. Since the angular velocity of the particle is given, it is convenient to express x and y in terms of the angle φ between OP and the x-axis:

$$x = \cos\varphi, \qquad y = \sin\varphi, \qquad \varphi' = -3.$$

Here the angular velocity is negative, since the point moves clockwise. The distance d between the points $P(x, y)$ and $Q(0, 2)$ satisfies the following relation:

$$d^2 = (x - 0)^2 + (y - 2)^2 = x^2 + y^2 - 4y + 4 = 1 - 4y + 4 = 5 - 4y.$$

Expressing y in terms of φ, we obtain the **basic relation** between φ and d:

$$d^2 = 5 - 4\sin\varphi.$$

Step 2 (differentiating the basic relation with respect to time t):

$$2dd' = -4(\cos\varphi)\varphi', \qquad d' = -2\frac{\cos\varphi}{d}\varphi'.$$

Step 3 (evaluating at time $t = t_0$).
We want to find d' at a specific time t_0 when $P(1, 0)$. So, at this time, we have $\varphi|_{t_0} = 0$ and $d|_{t_0} = \sqrt{5 - 4\sin 0} = \sqrt{5}$. Since $\varphi' = -3$, we get

$$d'|_{t_0} = -2\frac{\cos 0}{\sqrt{5}}(-3) = \frac{6}{\sqrt{5}}.$$

Answer: $\frac{6}{\sqrt{5}}\,m/s$.

5. Let $f(x) = x^3 + 3x^2 - 9x + 9$.
(a) $f'(x) = 3x^2 + 6x - 9$, $f''(x) = 6x + 6 = 6(x + 1)$. The **second derivative test** for concavity:

x	$(-\infty, -1)$	-1	$(-1, \infty)$
f''	$-$	0	$+$
f	$\frown$	inflexion point	$\smile$

How to memorize this rule? If $f''(x)$ is *positive*, "we are happy", so the graph of $y = f(x)$ is concave up (as a smile). If $f''(x)$ is *negative*, "we are unhappy", so the graph of $y = f(x)$ is concave down (as a frown).

Answer: f is concave down on $(-\infty, -1)$, and concave up on $(-1, \infty)$.

(b) In the above table, we found that f has inflexion point at $x = -1$ (since the function has opposite concavities on the left hand side and right hand side of this point). The y-coordinate of this point is $f(-1) = -1 + 3 - 9(-1) + 9 = -1 + 3 + 9 + 9 = 20$, so the **inflexion point** is $(-1, 20)$.

Let us find local extrema (i.e. local maxima and local minima) of f. Since f is a polynomial, it is differentiable everywhere, so it has no singular points. We also have no endpoints (since the domain of f is the whole real line). So, all local extrema of f must be critical points. To find critical points of f, we must solve the equation $f'(x) = 0$ for x:

$$3x^2 + 6x - 9 = 0, \qquad x^2 + 2x - 3 = 0, \qquad (x + 3)(x - 1) = 0.$$

So, the critical points of f are $x = -3$ and $x = 1$.

To establish the types of these critical points, we use the **second derivative test**: $f''(-3) = 6(-3 + 1) = -12 < 0$ and $f''(1) = 6(1 + 1) = 12 > 0$, so

x	-3	1
$f'(x)$	0	0
$f''(x)$	$-$	$+$
type	local max	local min

The y-coordinates at these points are $f(-3) = -27 + 27 + 27 + 9 = 36$ and $f(1) = 1 + 3 - 9 + 9 = 4$, so the **local minimum** point is $(1, 4)$ and the **local maximum** point is $(-3, 36)$.

Answer: The graph of $y = f(x)$ has the inflexion point $(-1, 20)$, the point of local minimum $(1, 4)$, and the point of local maximum $(-3, 36)$.

6. Solution: To sketch the graph of $y = f(x)$, we must find

1) the domain of f, and all vertical and horizontal, or diagonal (oblique) asymptotes (draw a small part of the graph near each asymptote!);

2) symmetries of the grapf, i.e. whether the function f is even or odd or neither;

3) intersection points of the graph of $y = f(x)$ with the coordinate axes;

4) intervals where f is increasing or decreasing, and points of local minima and local maxima of f;

6) intervals where f is concave up or concave down, and inflexion points of f.

Sketching the graph: Indicate the intercepts and the points of local maxima and minima, and join them, by monotone curves, with the parts of the graph (drawn in part 1) near the asymptotes. **Make sure that no additional intercepts appear, and the intervals where f is increasing or decreasing are exactly those found in part 4!** Indicate the inflexion points on the graph.

(a) $y = f(x) = \frac{x}{x^2-1} = \frac{x}{(x-1)(x+1)}$.

Domain of f and asymptotes: the function f is not defined at the points $x = -1$ and $x = 1$, so $D_f = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$, and the **vertical asymptotes** are $x = -1$ and $x = 1$. Moreover,

$$\lim_{x\to -1^-} f(x) = -\infty, \qquad \lim_{x\to -1^+} f(x) = \infty, \qquad \lim_{x\to 1^-} f(x) = -\infty, \qquad \lim_{x\to 1^+} f(x) = \infty.$$

Thus the graph approaches each of the asymptotes $x = -1$ and $x = 1$ from the left as $y \to -\infty$ and from the right as $y \to \infty$.

To find the **horizontal asymptotes**, let us find the limits of $f(x)$ as $x \to -\infty$ and $x \to \infty$:

$$\lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} \frac{x}{x^2-1} = \lim_{x\to -\infty} \frac{\frac{1}{x}}{1-\frac{1}{x^2}} = \frac{0^-}{1-0} = 0^-,$$

$$\lim_{x\to \infty} f(x) = \lim_{x\to \infty} \frac{x}{x^2-1} = \lim_{x\to \infty} \frac{\frac{1}{x}}{1-\frac{1}{x^2}} = \frac{0^+}{1-0} = 0^+,$$

so the **horizontal asymptote** is $y = 0$, and the graph approaches this asymptote from below as $x \to -\infty$ and from above as $x \to \infty$.

Since the horizontal asymptote is two-sided, there are no **diagonal (oblique) asymptotes**.

Now draw a small part of the graph near each asymptote!

Symmetries of the graph of $y = f(x)$:

$$f(-x) = \frac{-x}{(-x)^2-1} = -\frac{x}{x^2-1} = -f(x),$$

so the function f is **odd**, so the graph is **symmetric in the origin**, thus it is enough to draw the graph for $x \geq 0$ only, and then to reflect it in the origin.

Intersection points of the graph of $y = f(x)$ with coordinate axes: since $f(0) = 0$, the intersection point with the y-axis is $(0, 0)$. To find the intersection point with the x-axis, we solve the equation $f(x) = 0$ for x, so $x = 0$, and we get the same point $(0, 0)$.

Intervals where f is **increasing or decreasing**, and **local extreme points:**

$$y' = \frac{x^2 - 1 - 2x^2}{(x^2-1)^2} = \frac{-(x^2+1)}{(x-1)^2(x+1)^2} < 0$$

is negative at each point of the domain of f, so f **decreases** on each interval of its domain by the **first derivative test**:

x	$(-\infty,-1)$	-1	$(-1,1)$	1	$(1,\infty)$
$f'(x)$	$-$		$-$		$-$
f	$\searrow$	outside domain	$\searrow$	outside domain	$\searrow$

Since f is decreasing on each interval of its domain (and we do not have any endpoints), there are **no** points of **local minima** and **local maxima**.

Concavity and inflexion points (i.e. points where the concavity changes to the opposite). Since

$$y'' = -\left(\frac{x^2+1}{(x^2-1)^2}\right)' = -\frac{2x(x^2-1)^2 - 2(x^2-1)2x(x^2+1)}{(x^2-1)^4}$$

$$= -\frac{2x(x^2-1-2(x^2+1))}{(x^2-1)^3} = -\frac{2x(x^2-1-2x^2-2)}{(x^2-1)^3} = \frac{2x(x^2+3)}{(x-1)^3(x+1)^3},$$

x	$(-\infty,-1)$	-1	$(-1,0)$	0	$(0,1)$	1	$(1,\infty)$
$f''(x)$	$-$		$+$	0	$-$		$+$
f	$\frown$		$\smile$	inflexion point	$\frown$		$\smile$

Here the points $x = -1$ and $x = 1$ lie outside the domain of f. So, the only **inflexion point** is $(0,0)$.

Answer: horizontal asymptotes: $y = 0$ (two-sided),
vertical asymptotes: $x = -1$ and $x = 1$,
diagonal (oblique) asymptotes: none,
points of local maxima: none,
points of local minima: none,
inflexion points: $(0,0)$,
symmetries: graph is symmetric in the origin.

Now finish sketching the graph!

(b) $y = f(x) = \frac{x^2}{x^2+1}$.

Domain of f and asymptotes: the function f is defined for any $x \in \mathbb{R}$, so $D_f = (-\infty, \infty) = \mathbb{R}$. Since f is continuous on the whole $\mathbb{R}$, there are no vertical asymptotes. To find the **horizontal asymptotes**, let us find the limits of $f(x)$ as $x \to -\infty$ and $x \to \infty$:

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^2}{x^2+1} = \lim_{x\to-\infty} \frac{1}{1+\frac{1}{x^2}} = \frac{1}{1+0^+} = 1^-,$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2}{x^2+1} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x^2}} = \frac{1}{1+0^+} = 1^-,$$

so the **horizontal asymptote** is $y = 1$, and the graph approaches this asymptote from below as $x \to -\infty$ or $x \to \infty$.

Since the horizontal asymptote is two-sided, there are no **diagonal (oblique) asymptotes**.

Now draw a small part of the graph near each asymptote!

Symmetries of the graph of $y = f(x)$:

$$f(-x) = \frac{(-x)^2}{(-x)^2+1} = -\frac{x^2}{x^2+1} = f(x),$$

so the function f is **even**, so the graph is **symmetric in the y-axis**, thus it is enough to draw the graph for $x \geq 0$ only, and then to reflect it in the y-axis.

Intersection points of the graph of $y = f(x)$ with coordinate axes: since $f(0) = 0$, the intersection point with the y-axis is $(0, 0)$. To find the intersection point with the x-axis, we solve the equation $f(x) = 0$ for x, so $x = 0$, and we get the same point $(0, 0)$.

Intervals where f is **increasing or decreasing**, and **local extreme points:**

$$y' = \frac{2x(x^2+1) - 2x^3}{(x^2+1)^2} = \frac{2x}{(x^2+1)^2},$$

and by the **first derivative test** we have:

x	$(-\infty, 0)$	0	$(0, \infty)$
$f'(x)$	$-$	0	$+$
f	$\searrow$	local min	$\nearrow$

So, the point of **local minimum** is $(0, 0)$, and there are no points of **local maxima**.

Concavity and inflexion points (i.e. points where the concavity changes to the opposite). Since

$$y'' = \left(\frac{2x}{(x^2+1)^2}\right)' = \frac{2(x^2+1)^2 - 2(x^2+1)2x2x}{(x^2+1)^4}$$

$$= 2\frac{1-3x^2}{(x^2+1)^3} = 2\frac{(1-\sqrt{3}x)(1+\sqrt{3}x)}{(x^2+1)^3},$$

x	$(-\infty, -\frac{1}{\sqrt{3}})$	$-\frac{1}{\sqrt{3}}$	$(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$	$\frac{1}{\sqrt{3}}$	$(\frac{1}{\sqrt{3}}, \infty)$
$f''(x)$	$-$	0	$+$	0	$-$
f	$\frown$	inflexion point	$\smile$	inflexion point	$\frown$

So, the **inflexion points** are $(-\frac{1}{\sqrt{3}}, \frac{1}{4})$ and $(\frac{1}{\sqrt{3}}, \frac{1}{4})$.

Answer: horizontal asymptotes: $y = 1$ (two-sided),
vertical asymptotes: none,
diagonal (oblique) asymptotes: none,
points of local maxima: none,
points of local minima: $(0, 0)$,
inflexion points: $(-\frac{1}{\sqrt{3}}, \frac{1}{4})$ and $(\frac{1}{\sqrt{3}}, \frac{1}{4})$,
symmetries: graph is symmetric in y-axis.

Now finish sketching the graph!

(c) $y = f(x) = \frac{x^2-4}{x+1} = \frac{(x-2)(x+2)}{x+1}$.

Domain of f and asymptotes: the function f is not defined at the point $x = -1$, so $D_f = (-\infty, -1) \cup (-1, \infty)$, and the **vertical asymptote** is $x = -1$. Moreover,

$$\lim_{x \to -1^-} f(x) = \infty, \qquad \lim_{x \to -1^+} f(x) = -\infty.$$

Thus the graph approaches the asymptote $x = -1$ from the left as $y \to \infty$ and from the right as $y \to -\infty$.

To find out whether **horizontal asymptotes** exist, let us find the limits of $f(x)$ as $x \to -\infty$ and $x \to \infty$:

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{x^2 - 4}{x + 1} = \lim_{x \to -\infty} \frac{x - \frac{4}{x}}{1 + \frac{1}{x}} = \frac{-\infty - 0}{1 + 0} = -\infty, \qquad \text{DNE}$$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{x^2 - 4}{x + 1} = \lim_{x \to -\infty} \frac{x - \frac{4}{x}}{1 + \frac{1}{x}} = \frac{\infty - 0}{1 + 0} = \infty, \qquad \text{DNE}$$

so there are no **horizontal asymptotes**. But f may still have a **diagonal (oblique) asymptote** $y = ax+b$, such that $\lim_{x\to-\infty}(f(x)-(ax+b)) = 0$ or $\lim_{x\to\infty}(f(x)-(ax+b)) = 0$.

There are two ways to find the coefficients a, b of a diagonal (oblique) asymptote $y = ax + b$:

METHOD 1:

$$\lim_{x \to -\infty}(f(x) - (ax+b)) = \lim_{x \to -\infty}(\frac{x^2 - 4}{x + 1} - (ax+b)) = \lim_{x \to -\infty} \frac{x^2 - 4 - (x+1)(ax+b)}{x + 1}$$

$$= \lim_{x \to -\infty} \frac{(1-a)x^2 - (a+b)x - (b+4)}{x+1} = \lim_{x \to -\infty} \frac{(1-a)x - (a+b) - \frac{b+4}{x}}{1 + \frac{1}{x}}$$

$$= \lim_{x \to -\infty} \frac{(1-a)x - (a+b) - 0}{1+0} = \lim_{x \to -\infty}((1-a)x - (a+b)).$$

In order to get this limit being equal to 0, we must have $a = 1$ and $b = -1$, so we have a **diagonal (oblique) asymptote** $y = x - 1$ for $x \to -\infty$. For $x \to \infty$ we get the same a, b:

$$\lim_{x \to \infty}(f(x) - (ax+b)) = \lim_{x \to \infty}(\frac{x^2 - 4}{x + 1} - (ax+b)) = \lim_{x \to \infty} \frac{x^2 - 4 - (x+1)(ax+b)}{x + 1}$$

$$= \lim_{x\to\infty} \frac{(1-a)x^2-(a+b)x-(b+4)}{x+1} = \lim_{x\to\infty} \frac{(1-a)x-(a+b)-\frac{b+4}{x}}{1+\frac{1}{x}}$$
$$= \lim_{x\to\infty} \frac{(1-a)x-(a+b)-0}{1+0} = \lim_{x\to\infty}((1-a)x-(a+b)),$$
so $a=1$ and $b=-1$.

METHOD 2: The numbers a, b (if they exist) can be found by the formulae

$$a = \lim_{x\to-\infty} \frac{f(x)}{x}, \qquad b = \lim_{x\to-\infty}(f(x)-ax)$$

(or similarly with $x\to\infty$). So

$$a = \lim_{x\to-\infty} \frac{f(x)}{x} = \lim_{x\to-\infty} \frac{x^2-4}{x^2+x} = \lim_{x\to-\infty} \frac{1-\frac{4}{x^2}}{1+\frac{1}{x}} = \frac{1-0}{1+0} = 1,$$

$$b = \lim_{x\to-\infty}(f(x)-ax) = \lim_{x\to-\infty}(\frac{x^2-4}{x+1}-x) = \lim_{x\to-\infty} \frac{x^2-4-x^2-x}{x+1}$$
$$= \lim_{x\to-\infty} \frac{-4-x}{x+1} = \lim_{x\to-\infty} \frac{-\frac{4}{x}-1}{1+\frac{1}{x}} = \frac{0-1}{1+0} = -1,$$

so $y=x-1$ is the diagonal (oblique) asymptote of f for $x\to-\infty$. For $x\to\infty$, we get the same limits: $a=1$ and $b=-1$. (Check this!)

So $y=f(x)$ has the two-sided **diagonal (oblique) asymptote** $y=x-1$. Moreover, $\lim_{x\to-\infty}(f(x)-(x-1)) = \frac{0^+}{1+0} = 0^+$ and $\lim_{x\to\infty}(f(x)-(x-1)) = \frac{0^-}{1+0} = 0^-$. Thus the graph approaches this slant asymptote from above as $x\to-\infty$, and from below as $x\to\infty$.

Now draw a small part of the graph near each asymptote!

Symmetries of the graph of $y=f(x)$:

$$f(-x) = \frac{(-x)^2-4}{-x+1} = \frac{x^2-4}{-x+1}.$$

This function equals neither f nor $-f$. So the function f is neither **even** nor **odd**, so we do not expect any symmetry of its graph. (Actually, the graph is symmetric in the point $(-1,-2)$.)

Intersection points of the graph of $y=f(x)$ with coordinate axes: since $f(0)=-4$, the intersection point with the y-axis is $(0,-4)$. To find the intersection point with the x-axis, we solve the equation $f(x)=0$ for x, so $x=2$ or $x=-2$, and we get two points $(2,0)$ and $(-2,0)$.

Intervals where f is **increasing or decreasing**, and **local extreme points:**

$$y' = \left(\frac{x^2-4}{x+1}\right)' = \frac{2x(x+1)-(x^2-4)}{(x+1)^2} = \frac{2x^2+2x-x^2+4}{(x+1)^2} = \frac{x^2+2x+4}{(x+1)^2} = \frac{(x+1)^2+3}{(x+1)^2} > 0,$$

is positive at each point of the domain of f, so f **increases** on each interval of its domain by the **first derivative test**:

x	$(-\infty,-1)$	-1	$(-1,\infty)$
$f'(x)$	$+$		$+$
f	$\nearrow$	outside domain	$\nearrow$

Since f is increasing on each interval of its domain (and we do not have any endpoints), there are no points of **local minima** and **local maxima**.

Concavity and inflexion points (i.e. points where the concavity changes to the opposite). Since

$$y'' = \left(\frac{x^2+2x+4}{(x+1)^2}\right)' = \frac{(2x+2)(x+1)^2 - 2(x+1)(x^2+2x+4)}{(x+1)^4}$$

$$= 2\frac{(x+1)^2 - (x^2+2x+4)}{(x+1)^3} = 2\frac{-3}{(x+1)^3} = \frac{-6}{(x+1)^3},$$

x	$(-\infty, -1)$	-1	$(-1, \infty)$
$f''(x)$	$+$		$-$
f	$\smile$	outside domain	$\frown$

So, the only inflexion point can be $x = -1$, but this point does not belong to the domain of f. Therefore, there are **no inflexion points**.

Answer: horizontal asymptotes: none,
vertical asymptotes: $x = -1$,
diagonal (oblique) asymptotes: $y = x - 1$ (two-sided),
points of local maxima: none,
points of local minima: none,
inflexion points: none,
symmetries: none. (Actually, the graph is symmetric in the point $(-1, -2)$.)

Now finish sketching the graph!

(d) $y = f(x) = \frac{\ln x}{x}$.

Domain of f and asymptotes: the function f is defined for any $x > 0$ (since $\ln x$ is defined for $x > 0$ only), so $D_f = (0, \infty)$. Since f is continuous in its domain, the only **vertical asymptote** may be $x = 0$ (the only endpoint of D_f), provided that $\lim_{x\to 0^+} f(x) = \pm\infty$. Now

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\ln x}{x} = \text{“}\frac{-\infty}{0^+}\text{”} = -\infty,$$

so $x = 1$ is the **vertical asymptote**, and the graph approaches this asymptote from the right as $y \to -\infty$.

To find **horizontal asymptotes**, let us find the limit of $f(x)$ as $x \to \infty$:

$$\lim_{x\to \infty} f(x) = \lim_{x\to \infty} \frac{\ln x}{x} = \text{“}\frac{\infty}{\infty}\text{”},$$

so both $\ln x$ and x become very large as $x \to \infty$, and we must compare the rates of their growths. As we know, the growth of $\ln x$ is slower than the growth of any polynomial, so slower than the growth of x. (Remember: “in a battle between $\ln x$ and any polynomial, the polynomial always wins”.) So, we have

$$= \text{“}\frac{1}{\infty}\text{”} = 0^+,$$

so the **horizontal asymptote** is $y = 0$, and the graph approaches this asymptote from above as $x \to \infty$.

Thus there are **no diagonal (oblique) asymptotes**.

Now draw a small part of the graph near each asymptote!

Symmetries of the graph of $y = f(x)$. The domain of f is $(0, \infty)$, so it is finite from the left and infinite from the right. So, the graph of f has **no symmetries**, since its domain itself is not symmetric.

Intersection points of the graph of $y = f(x)$ with coordinate axes: since the point $x = 0$ does not belong the domain, there are no intersections with the y-axis. To find intersection points with the x-axis, we solve the equation $f(x) = 0$ for x, so $\frac{\ln x}{x} = 0$, $\ln x = 0$, $x = 1$ (remember that the function $\ln x$ is increasing and intersects the y-axis at the point $x = 1$). So the **intersection point** with the x-axis is $(1, 0)$.

Intervals where f is **increasing or decreasing**, and **local extreme points:**

$$y' = \left(\frac{\ln x}{x}\right)' = \frac{\frac{1}{x}x - \ln x}{x^2} = \frac{1 - \ln x}{x^2},$$

and by the **first derivative test** we have:

x	$(0, e)$	e	(e, ∞)
$f'(x)$	$+$	0	$-$
f	$\nearrow$	local max	$\searrow$

So, the point of **local maximum** is $(e, \frac{1}{e})$, and there are no points of **local minima**.

Concavity and inflexion points (i.e. points where the concavity changes to the opposite). Since

$$y'' = \left(\frac{1-\ln x}{x^2}\right)' = \frac{-\frac{1}{x}x^2 - 2x(1-\ln x)}{x^4}$$
$$= \frac{-x-2x+2x\ln x}{x^4} = \frac{-3+2\ln x}{x^3},$$

x	$(0, e^{3/2})$	$e^{3/2}$	$(e^{3/2}, \infty)$
$f''(x)$	$-$	0	$+$
f	$\frown$	inflexion point	$\smile$

So, the **inflexion point** is $(e^{3/2}, \frac{3}{2e^{3/2}})$.

Answer: horizontal asymptotes: $y = 0$,
vertical asymptotes: $x = 0$,
diagonal (oblique) asymptotes: none,
points of local maxima: $(e, \frac{1}{e})$,
points of local minima: none,
inflexion points: $(e^{3/2}, \frac{3}{2e^{3/2}})$,
symmetries: none.

Now finish sketching the graph!

(e) $y = f(x) = x^2e^x \geq 0$.

Domain of f and asymptotes: the function f is defined for any $x \in \mathbb{R}$, so $D_f = (-\infty, \infty) = \mathbb{R}$. Since f is continuous on the whole $\mathbb{R}$, there are **no vertical asymptotes**.

To find the **horizontal asymptotes**, let us find the limits of $f(x)$ as $x \to -\infty$ and $x \to \infty$:

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} x^2e^x = \text{“}\infty \cdot 0^{+}\text{”} = 0^+,$$

since the exponential growth is much faster than the polynomial growth. (Remember that, in a battle between a polynomial and an exponential function, the exponential function always wins!)

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} x^2e^x = \text{“}\infty \cdot \infty\text{”} = \infty, \qquad \text{DNE},$$

so the only **horizontal asymptote** is $y = 0$ (for $x \to -\infty$). Moreover, the graph approaches this asymptote from above as $x \to -\infty$.

There still can be a **diagonal (oblique) asymptote** $y = ax + b$ for $x \to \infty$. If such an asymptote would exist, we would have

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} xe^x = \text{“}\infty \cdot \infty\text{”} = \infty, \qquad \text{DNE}.$$

So the function $f(x)$ approaches ∞ much faster than any diagonal line as $x \to \infty$, so there are **no diagonal (oblique) asymptotes**.

Now draw a small part of the graph near each asymptote!

Symmetries of the graph of $y = f(x)$:

$$f(-x) = (-x)^2 e^{-x} = x^2 e^{-x}.$$

This function equals neither f nor $-f$. So the function f is neither **even** nor **odd**, so we do not expect any symmetry of its graph.

Intersection points of the graph of $y = f(x)$ with coordinate axes: since $f(0) = 0$, the intersection point with the y-axis is $(0, 0)$. To find the intersection point with the x-axis, we solve the equation $f(x) = 0$ for x, so $x = 0$ (since $e^x > 0$ for any x), and we get the same point $(0, 0)$.

Intervals where f is **increasing or decreasing**, and **local extreme points:**

$$y' = \left(x^2 e^x\right)' = 2xe^x + x^2 e^x = (x^2 + 2x)e^x = x(x+2)e^x,$$

and by the **first derivative test** we have:

x	$(-\infty, -2)$	-2	$(-2, 0)$	0	$(0, \infty)$
$f'(x)$	$+$		$-$		$+$
f	$\nearrow$	local max	$\searrow$	local min	$\nearrow$

So, the point of **local maximum** is $(-2, \frac{4}{e^2})$, and the point of **local minimum** is $(0, 0)$.

Concavity and inflexion points (i.e. points where the concavity changes to the opposite). Since

$$y'' = \left((x^2 + 2x)e^x\right)' = (2x+2)e^x + (x^2+2x)e^x = (x^2+4x+2)e^x = (x-(-2+\sqrt{2}))(x-(-2-\sqrt{2}))e^x,$$

x	$(-\infty, -2-\sqrt{2})$	$-2-\sqrt{2}$	$(-2-\sqrt{2}, -2+\sqrt{2})$	$-2+\sqrt{2}$	$(-2+\sqrt{2}, \infty)$
$f''(x)$	$+$	0	$-$	0	$+$
f	$\smile$	inflexion point	$\frown$	inflexion point	$\smile$

So, the **inflexion points** are $(-2-\sqrt{2}, \frac{6+4\sqrt{2}}{e^{2+\sqrt{2}}})$ and $(-2+\sqrt{2}, \frac{6-4\sqrt{2}}{e^{2-\sqrt{2}}})$.

Answer: horizontal asymptotes: $y = 0$ (for $x \to -\infty$),
vertical asymptotes: none,
diagonal (oblique) asymptotes: none,
points of local maxima: $(-2, \frac{4}{e^2})$,
points of local minima: $(0, 0)$,
inflexion points: $(-2-\sqrt{2}, \frac{6+4\sqrt{2}}{e^{2+\sqrt{2}}})$ and $(-2+\sqrt{2}, \frac{6-4\sqrt{2}}{e^{2-\sqrt{2}}})$.
symmetries: none.

Now finish sketching the graph!

7. (a) Among all rectangles of given area, show that the square has the least perimeter.

Solution: We divide the solution into three steps.

Step 1 (reducing the question to an appropriate min problem).
A rectangle with sides x, y has perimeter $P = 2(x + y)$ and area $A = xy$. The area is given, so A is a given constant ($A > 0$), thus $y = \frac{A}{x}$ and we can express perimeter in terms of x only:

$$P(x) = 2(x + \frac{A}{x}).$$

We want to minimize the function $P(x)$ on the interval $x \in (0, \infty)$ (since $x > 0$ and $y = \frac{A}{x} > 0$).

Step 2 (solving the min problem).
Since the function $P(x) = 2(x + \frac{A}{x})$ is differentiable everywhere on its domain, it has **no singular points**. Let us find critical points of P. We must solve the equation $P'(x) = 0$ for x:

$$P'(x) = 2(1 - \frac{A}{x^2}) = 2\frac{x^2 - A}{x^2} = 2\frac{(x - \sqrt{A})(x + \sqrt{A})}{x^2},$$

so the only **critical point** is $x = \sqrt{A}$ (since the point $x = -\sqrt{A}$ does not belong to the domain). We can proceed with this step in three different ways.

METHOD 1: The function $P(x)$ is defined on the open interval $D_P = (0, \infty)$. Let us find the limits of $P(x)$ as x approach one of the endpoints of this interval:

$$\lim_{x\to 0^+} P(x) = \lim_{x\to 0^+} 2(x + \frac{A}{x}) = \text{“}0 + \infty\text{”} = \infty,$$

$$\lim_{x\to \infty} P(x) = \lim_{x\to \infty} 2(x + \frac{A}{x}) = \text{“}\infty + 0\text{”} = \infty.$$

Because of this, and since the function P is continuous on the interval $(0, \infty)$, it has an **absolute minimum** at some point $x_0 \in (0, \infty)$. The only candidates for the point x_0 are singular points and critical points. Since we do not have any singular points (see above), and the only critical point is $x = \sqrt{A}$, the function P has an **absolute minimum** at the point $x_0 = \sqrt{A}$.

METHOD 2 (perhaps the most suitable method for this question):
Let us use the **first derivative test**:

x	$(0, \sqrt{A})$	$\sqrt{A}$	$(\sqrt{A}, \infty)$
$P'(x)$	$-$	0	$+$
P	$\searrow$	local min	$\nearrow$

So, $P(x)$ has an **absolute minimum** at the critical point $x = \sqrt{A}$.

METHOD 3: Let us find the **type of the critical point** $x = \sqrt{A}$ of P. We use the **second derivative test**:

$$P''(x) = 2(1 - \frac{A}{x^2})' = \frac{4A}{x^3} > 0,$$

so $P(x)$ has a **local minimum** at the critical point $x = \sqrt{A}$. Actually, this local minimum must be an **absolute minimum**, since $x = \sqrt{A}$ is the only critical point, and there are no singular points (see above).

Step 3 (making sure to answer the question).
We are asked to show that the rectangle must be a square, in order to have the least perimeter. We found that the perimeter P has an **absolute minimum** at the point $x = \sqrt{A}$. For this value of x, we have

$$y = \frac{A}{x} = \frac{A}{\sqrt{A}} = \sqrt{A} = x,$$

so the rectangle must be a square.

(b) A window has perimeter $10m$ and is in the shape of a rectangle with the top edge replaced by a semicircle. Find the dimensions of the rectangle if the window admits the greatest amount of light.

Solution: We divide the solution into three steps.

Step 1 (reducing the question to an appropriate max problem).
Denote the sides of the rectangle by x (the horizontal side) and y (the vertical side), so $x > 0$ and $y > 0$. The semicircle has diameter x, so its radius is $r = \frac{x}{2}$, and its length is $\frac{2\pi r}{2} = \pi r = \frac{\pi x}{2}$. Thus the perimeter of the window equals

$$P = x + 2y + \frac{\pi}{2}x = 10, \quad 2y = 10 - (x + \frac{\pi}{2}x), \quad y = 5 - (1 + \frac{\pi}{4})x = 5 - \frac{4+\pi}{4}x.$$

The area of the window equals

$$A(x) = xy + \frac{\pi r^2}{2} = x(5 - \frac{4+\pi}{4}x) + \frac{\pi x^2}{8} = 5x - \frac{4+\pi}{4}x^2 + \frac{\pi}{8}x^2 = 5x - \frac{8+\pi}{8}x^2 = x(5 - \frac{8+\pi}{8}x).$$

We want to maximize the function $A(x)$ on the interval $x \in [0, \frac{20}{4+\pi}]$
(since $x \geq 0$ and $y = 5 - (1 + \frac{\pi}{4})x \geq 0$, so $0 \leq x \leq \frac{5}{1+\frac{\pi}{4}} = \frac{20}{4+\pi}$).

Step 2 (solving the max problem).
Since the function A is a polynomial, it is differentiable everywhere, so it has **no singular points**. Let us find critical points of A. We must solve the equation $A'(x) = 0$ for x:

$$A'(x) = 5 - \frac{8+\pi}{8}2x = 5 - \frac{8+\pi}{4}x,$$

so the only **critical point** is $x = \frac{20}{8+\pi}$. This point belongs the domain, since

$$0 < \frac{20}{8+\pi} < \frac{20}{4+\pi}.$$

We can proceed with this step in three different ways.

METHOD 1 (the algebra will be complicated here):
The function $A(x)$ is defined on the **closed** interval $D_A = [0, \frac{20}{4+\pi}]$. Since A is a polynomial, it is **continuous**. Therefore it must have an **absolute maximum** at some point $x_0 \in D_A = [0, \frac{20}{4+\pi}]$ (by the theorem about existance of an absolute maximum of a continuous function defined on a closed interval). The only candidates for the point x_0 are singular points of A, critical points of A, and the endpoints of D_A. There are **no singular**

points, see above. The only **critical point** is $x = \frac{20}{8+\pi}$, and the **endpoints** are $x = 0$ and $x = \frac{20}{4+\pi}$. Let us compare the values of $A(x) = x(5 - \frac{8+\pi}{8}x)$ at these points: $A(0) = 0$,

$$A\left(\frac{20}{8+\pi}\right) = \frac{20}{8+\pi}\cdot\left(5 - \frac{8+\pi}{8}\cdot\frac{20}{8+\pi}\right) = \frac{20}{8+\pi}\cdot\left(5 - \frac{5}{2}\right) = \frac{20}{8+\pi}\cdot\frac{5}{2} = \frac{50}{8+\pi} \approx 4.5,$$

$$A\left(\frac{20}{4+\pi}\right) = \frac{20}{4+\pi}\cdot\left(5 - \frac{8+\pi}{8}\cdot\frac{20}{4+\pi}\right) = \frac{20}{4+\pi}\cdot\left(5 - \frac{5}{2}\frac{8+\pi}{4+\pi}\right) = \frac{50\pi}{(4+\pi)^2} \approx 3,$$

so

x	0	$\frac{20}{8+\pi}$	$\frac{20}{4+\pi}$
$A(x)$	0	$\frac{50}{8+\pi} \approx 4.5$	$\frac{50\pi}{(4+\pi)^2} \approx 3$

We see that the value of $A(x)$ is maximal if $x = \frac{20}{8+\pi}$. So, the function A has an **absolute maximum** at the critical point $x = \frac{20}{8+\pi}$.

METHOD 2 (perhaps the most suitable method for this question):
We use the **first derivative test**:

x	$(0, \frac{20}{8+\pi})$	$\frac{20}{8+\pi}$	$(\frac{20}{8+\pi}, \frac{20}{4+\pi})$
$A'(x)$	$+$	0	$-$
A	$\nearrow$	local max	$\searrow$

So, the function A has an **absolute maximum** at the critical point $x = \frac{20}{8+\pi}$.

METHOD 3: Let us find the **type of the critical point** $x = \frac{20}{8+\pi}$. We use the **second derivative test**:

$$A''(x) = (5 - \frac{8+\pi}{4}x)' = 0 - \frac{8+\pi}{4} = -\frac{8+\pi}{4} < 0,$$

so $A(x)$ has a **local maximum** at the critical point $x = \frac{20}{8+\pi}$. Actually this local minimum must be an **absolute maximum**, since $x = \frac{20}{8+\pi}$ is the only critical point, and there are no singular points (see above).

Step 3 (making sure to answer the question).
We are asked to find the dimensions of the rectangle, in order to get the largest area of the window. We found that the area A has an **absolute maximum** at the point $x = \frac{20}{8+\pi}$. For this value of x, we have

$$y = 5-(1+\frac{\pi}{4})x = 5-\frac{5}{2+\frac{\pi}{4}} = 5(1-\frac{1}{2+\frac{\pi}{4}}) = 5(1-\frac{4}{8+\pi}) = 5\frac{8+\pi-4}{8+\pi} = 5\frac{4+\pi}{8+\pi}.$$

Answer: $x = \frac{20}{8+\pi}m$ and $y = 5\frac{4+\pi}{8+\pi}m$.

8. Let (a) $f(x) = x^4 - 8x^2 - x + 16$, $x_0 = 2$; (b) $f(x) = x^3 - 997$, $x_0 = 10$. Take the number x_0 as the first approximation for the root of $f(x) = 0$. Find the next approximation x_1, using Newton's Method for Approximating Roots.

Solution: By **Newton's method** for approximating roots, we have $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, thus $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$.

(a) Since $x_0 = 2$, we have

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)} \ .$$

Now $f(2) = 16 - 32 - 2 + 16 = -2$, $f'(x) = 4x^3 - 16x - 1$, so $f'(2) = 32 - 32 - 1 = -1$. Thus

$$x_1 = 2 - \frac{-2}{-1} = 2 - 2 = 0.$$

Answer: $x_1 = 0$.

(b) Since $x_0 = 10$, we have

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 10 - \frac{f(10)}{f'(10)} \ .$$

Now $f(10) = 1000 - 997 = 3$, $f'(x) = 3x^2$, so $f'(10) = 300$. Thus

$$x_1 = 10 - \frac{3}{300} = 10 - \frac{1}{100} = 9.99$$

Answer: $x_1 = 9.99$

9. (a) Approximate $\sqrt[3]{997}$ using linearization.
(b) Determine the sign of the Error, and estimate its size.
(c) Specify an interval you can be sure contains $\sqrt[3]{997}$.

Solution:

(a) The part (a) coincides with question 4 from the midterm.

Since $\sqrt[3]{1000} = 10$, it is useful to approximate the function $y = f(x) = \sqrt[3]{x}$ near the point $a = 1000$. Let us approximate the graph of $y = f(x)$ by its tangent line at the point $(a, f(a)) = (1000, 10)$. The equation of this tangent line is

$$y = L(x) \qquad \text{where} \qquad L(x) = f(a) + f'(a) \cdot (x - a).$$

The function $L(x)$ is called the *linear approximation* of $f(x)$ at the point a. Now

$$f'(x) = \frac{1}{3\sqrt[3]{x^2}},$$

so

$$L(x) = \sqrt[3]{1000} + \frac{x - 1000}{3\sqrt[3]{1000^2}} = \sqrt[3]{1000} + \frac{x - 1000}{3(\sqrt[3]{1000})^2} = 10 + \frac{x - 1000}{300}.$$

Since $f(x) \approx L(x)$ if the point $x = 997$ is close enough to $a = 1000$, we have

$$\sqrt[3]{997} = f(997) \approx L(997) = 10 + \frac{-3}{300} = 10 - \frac{1}{100} = 9.99$$

Answer: $\sqrt[3]{997} \approx 9.99$ (We got the same approximation as in question 8(b) where we used one step of Newton's method!)

(b) We use the following formula for the Error $E(x) = f(x) - L(x)$ of the linear approximation:

$$E(x) = \frac{f''(c)}{2}(x - a)^2 \qquad \text{for some } c \in (x, a).$$

Here we used that $x = 997 < 1000 = a$. Now

$$f'(x) = (\sqrt[3]{x})' = \left(x^{1/3}\right)' = \frac{1}{3}x^{-2/3},$$

$$f''(x) = \left(\frac{1}{3}x^{-2/3}\right)' = \frac{1}{3}(-\frac{2}{3})x^{-5/3} = -\frac{2}{9x^{5/3}} < 0,$$

therefore

$$E(997) = \frac{f''(c)}{2}(997 - 1000)^2 = -\frac{3^2}{9c^{5/3}} = -\frac{1}{c^{5/3}} < 0.$$

Let us estimate the absolute value of the Error which is

$$|E(997)| = \frac{1}{c^{5/3}}$$

We do not know the exact value of c, but we know that $c \in (997, 1000)$. So, let us try to maximize our expression if $c \in (997, 1000)$:

$$< \frac{1}{997^{5/3}}$$

Here we replaced c by 997, which is the smallest possible value of c. Actually, since we want to take the $\frac{5}{3}$th power of this number, it is more suitable to replace c by a number which is a perfect cube, and is even less than 997. So, let us replace c by $9^3 = 729 < 997$:

$$< \frac{1}{729^{5/3}} = \frac{1}{(9^3)^{5/3}} = \frac{1}{9^5} = \frac{1}{59049} < \frac{1}{50000} = .00002\ .$$

Answer: $E(997) < 0$, $|E(997)| < .00002$.

(c) By (b), the Error of our approximation is negative, so $\sqrt[3]{997} < 9.99$. Also by (b), the absolute value of Error is less than $.00002$, so $\sqrt[3]{997} > 9.99 - .00002 = 9.98998$. Therefore $9.98998 < \sqrt[3]{997} < 9.99$.

Answer: $\sqrt[3]{997} \in (9.98998, 9.99)$.

Remark 1: By (b), [ERROR]=[TRUE VALUE]$-$[ESTIMATE]< 0, so [TRUE VALUE]<[ESTIMATE], thus our estimate 9.99 is an **overestimate**. From (c) we see that 9.98998 is an **underestimate** for $\sqrt[3]{997}$.

Remark 2: We see from (c) that the first 5 decimals of $\sqrt[3]{997}$ are 9.9899, and the 6th decimal can be 8 or 9:

either $\sqrt[3]{997} = 9.98998...$ or $\sqrt[3]{997} = 9.98999...$

Actually $\sqrt[3]{997}$ is irrational, so its decimal expansion is neither terminating nor periodic.