MATH 251
WORKSHEET #6 – SOLUTIONS

1. Calculate the Taylor polynomial of degree 3 for the following functions:

$$\text{(a)}\ \ f(x) = \frac{1}{x} \quad \text{about } x = 1,$$

$$\text{(b)}\ \ f(x) = x^5 - x^4 + 2x^3 + 6x - 1 \quad \text{about } x = 0.$$

(a) **Solution:** We need 3 derivatives of f at $x = 1$:

$$f'(x) = -\frac{1}{x^2}, \qquad f''(x) = \frac{2}{x^3}, \qquad f'''(x) = -\frac{6}{x^4},$$

now evaluate at $x = 1$:

$$f(1) = 1, \qquad f'(1) = -1, \qquad f''(1) = 2, \qquad f'''(1) = -6,$$

so

$$P_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$$

$$= 1 - 1(x-1) + \frac{2}{2!}(x-1)^2 + \frac{-6}{3!}(x-1)^3 = 1 - (x-1) + (x-1)^2 - (x-1)^3.$$

(b) **Answer:** $P_3(x) = -1 + 6x + 2x^3$.

Solution: This can be done in the same way as for (a), by taking the first 3 derivatives and evaluating them at $t = 0$. However, if one understands that the Taylor polynomial of degree k, by definition, is the polynomial P_k of degree k that best approximates f near the point $x_0 = 0$, then it is clear that when f is itself a polynomial, $P_k(x)$ should equal $f(x)$ itself, at least through degree k. Thus the best fitting polynomial $P_3(x)$ to $f(x)$ will simply be $2x^3 + 6x - 1$. To prove this fact, notice that the 1'st, 2'nd, 3'rd derivatives of the terms $x^5 - x^4$ all still contain some positive power of x, hence when $x = 0$ these evaluate to 0. So only the terms $2x^3 + 6x - 1$ can affect the answer, and obviously the best fitting degree 3 polynomial to this is it itself, $2x^3 + 6x - 1$.

2. Find the following limits:

(a) $$\lim_{x\to 0} \frac{3^x - 1}{x} = \text{“}\frac{1-1}{0}\text{”} = \text{“}\frac{0}{0}\text{”} = \text{(by l'Hôpital's Rule)} = \lim_{x\to 0} \frac{(3^x - 1)'}{(x)'}$$

In order to find $(3^x)'$, we use **logarithmic differentiation**: $y = 3^x$, $\ln y = x \ln 3$, $\frac{y'}{y} = \ln 3$, $y' = y \ln 3 = 3^x \ln 3$, so

$$= \lim_{x\to 0} \frac{3^x \ln 3 - 0}{1} = \ln 3.$$

(b) $$\lim_{x\to 1^-} \frac{\arccos x}{\sqrt{1-x}} = \text{“}\frac{\arccos 1}{1-1}\text{”} = \text{“}\frac{0}{0}\text{”} = \text{(by l'Hôpital's Rule)} = \lim_{x\to 1^-} \frac{-\frac{1}{\sqrt{1-x^2}}}{-\frac{1}{2\sqrt{1-x}}}$$

$$= \lim_{x\to 1^-} \frac{2\sqrt{1-x}}{\sqrt{1-x^2}} = \lim_{x\to 1^-} \frac{2\sqrt{1-x}}{\sqrt{(1-x)(1+x)}} = \lim_{x\to 1^-} \frac{2}{\sqrt{1+x}} = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

(c)
$$\lim_{x\to \pi/2} \frac{\sin t}{t} = \frac{\sin(\pi/2)}{\pi/2} = \frac{1}{\pi/2} = \frac{2}{\pi},$$

here the limit has **determinate type**, so we are **not allowed** to apply the l'Hôpital's Rule.

(d)
$$\lim_{x\to 1} x^{\frac{1}{x-1}} = \text{“}1^{\infty}\text{”},$$

so the limit is **indeterminate**. We will find this limit by taking “ $\ln$ ”:

$$\ln(\lim_{x\to 1} x^{\frac{1}{x-1}}) = \lim_{x\to 1} \ln(x^{\frac{1}{x-1}}) = \lim_{x\to 1} \frac{\ln x}{x-1} = \text{“}\frac{0}{0}\text{”}$$

$$= \text{(by l'Hôpital's Rule)} = \lim_{x\to 1} \frac{\frac{1}{x}}{1} = \lim_{x\to 1} \frac{1}{x} = \frac{1}{1} = 1,$$

and do not forget to **take the exponent**:

$$\lim_{x\to 1} x^{\frac{1}{x-1}} = e^1 = e.$$

(e)
$$\lim_{x\to -\infty} \frac{2x+\sin(3x)}{x-\pi} = \text{“}\frac{-\infty}{\infty}\text{”}$$

an **indefinite limit**, so we can proceed in two different ways:

$$= \lim_{x\to -\infty} \frac{2+\frac{\sin(3x)}{x}}{1-\frac{\pi}{x}}$$

Let us find $\lim_{x\to -\infty} \frac{\sin(3x)}{x}$. Since $\frac{1}{x} \leq \frac{\sin(3x)}{x} \leq -\frac{1}{x}$ for $x < 0$, and both “squeezing functions” approach 0 as $x \to -\infty$, our function $\frac{\sin(3x)}{x}$ is forced to approach the same limit **by the Squeeze Theorem**. So $\lim_{x\to -\infty} \frac{\sin(3x)}{x} = 0$ and

$$= \frac{2+0}{1-0} = \frac{2}{1} = 2.$$

Another method: Let us try to apply the l'Hôpital's Rule:

$$\lim_{x\to -\infty} \frac{2+3\cos(3x)}{1} = \lim_{x\to -\infty} (2+3\cos(3x)) \qquad \mathbf{DNE},$$

since the function $y = 2 + 3\cos(3x)$ keeps oscillating between $y = -1$ and $y = 5$ as $x \to -\infty$. Since this limit does not exist, we **can not** apply the l'Hôpital's Rule. So, the **l'Hôpital's Rule fails** in this example.

3. Write the following sums using sigma notation:

(a) $$5+6+7+8+9=\sum_{j=5}^{9} j \qquad \textbf{OR} \qquad 5+6+7+8+9=\sum_{j=1}^{5}(j+4).$$

(b) $$1-x+x^2-x^3+\ldots-x^{49}+x^{50}=\sum_{j=0}^{50}(-1)^j x^j.$$

(c) $$\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\ldots+\frac{n}{2^n}=\sum_{j=1}^{n}\frac{j}{2^j}.$$

4. Evaluate

(a) $$\sum_{j=0}^{8}(2^{j+1}-2j)=(\sum_{j=0}^{8}2^{j+1})-2(\sum_{j=0}^{8}j)=(\sum_{j=1}^{9}2^j)-2(\sum_{j=1}^{8}j)$$

$$=\frac{2^{9+1}-2}{2-1}-2\frac{8(8+1)}{2}=(2^{10}-2)-8\cdot 9=1024-2-72=950.$$

Here we found the sum $\sum_{j=1}^{9}2^j=2+2^2+2^3+2^4+\ldots+2^8+2^9$ using the method for finding the **sum of geometric progression**: $(x+x^2+x^3+x^4+\ldots+x^8+x^9)(x-1)=x^{9+1}-x=x^{10}-x,$ so $x+x^2+x^3+x^4+\ldots+x^8+x^9=\frac{x^{10}-x}{x-1}$.

(b) $$\sum_{j=1}^{50}\frac{1}{j(j+1)}=\sum_{j=1}^{50}\left(\frac{1}{j}-\frac{1}{j+1}\right)$$

$$=(\frac{1}{1}-\frac{1}{2})+(\frac{1}{2}-\frac{1}{3})+(\frac{1}{3}-\frac{1}{4})+\ldots+(\frac{1}{49}-\frac{1}{50})+(\frac{1}{50}-\frac{1}{51})$$

this is a “telescoping sum” (since all terms except the first and the last terms cancel), so we have

$$=\frac{1}{1}-\frac{1}{51}=\frac{51-1}{51}=\frac{50}{51}.$$