Practice Problems S7

1. Given

$$f(x) = \frac{x^2-9}{x^2-4}, \quad f'(x) = \frac{10x}{(x-2)^2(x+2)^2},$$
$$f''(x) = \frac{-10(3x^2+4)}{(x-2)^3(x+2)^3}, \quad \lim_{x\to-\infty} f(x) = 1 = \lim_{x\to+\infty} f(x),$$
$$\lim_{x\to-2^-} f(x) = -\infty = \lim_{x\to2^+} f(x), \quad \lim_{x\to-2^+} f(x) = +\infty = \lim_{x\to2^-} f(x),$$

sketch the graph of f (Highlight all asymptotes, intercepts, relative extrema and inflection points if any).

2. Find all critical points, the absolute maximum and absolute minimum values of $f(x)$ and state where they occur:

   (a) $f(x) = (x^2+x)^{\frac{2}{3}}$ on $[-2, 3]$;

   (b) $f(x) = 5\sqrt{x^2+9} - 4x$ on $[0, 5]$.

3. Find all relative extrema of the following fonctions:

   (a) $(x) = (x^2+x)^{\frac{2}{3}}$;

   (b) $f(x) = x^3 - 3x^2 - 9x$.

4. Use Rolle's Theorem to prove that $x^3 + 3x + 1 = 0$ has exactly one real root.

5. Determine whether the Mean-Value Theorem (MVT) can be applied to the function $f(x)$ on the closed interval $[a, b]$. If yes, find all points between a and b where the instantaneous rate of change matches the average rate of change of $f(x)$ over $[a, b]$. Otherwise, explain your answer:

(a) $f(x) = x\ln(x)$ on $[2, 4]$;

(b) $f(x) = \sqrt{2-x}$ on $[-7, 2]$

(c) $f(x) = x^{\frac{2}{3}}$ on $[-1, 3]$.

6. If the radius r of a sphere is increasing at a rate of 2 cm per minute. Find the rate of change of the volume when $r = 6$ cm.

7. As the sun sets behind a 120-foot building, the building's shadow grows. How fast is the shadow growing (in feet per minute) when the sun's rays make an angle of $\pi/4$ radians?

8. A rectangle is to be inscribed in a right triangle having sides 6 cm, 8 cm and 10 cm long. If two sides of the rectangle lie on the sides of the right angle of the triangle, find the dimensions of the rectangle with the greatest area.

9. A cylindrical can, open at the top, is to hold 500 cm^3 of beer. Find the height and the radius that minimize the amount of material needed to manufacture the can.

10. An open box is to be made from a 3 cm by 8 cm rectangular piece of sheet metal by cutting out squares of equal size from the four corners and bending up the sizes. Find the maximum volume that the open box can have.

11. A rectangular area of 3200 m^2 is to be fenced off. Two opposite sides will use a fencing costing \$1 per meter and the remaining sides will use fencing costing \$2 per meter. Find the dimensions of the rectangle of least cost.

12. A rectangle has its lower corners on the x-axis and its two upper corners on the curve $y = 16 - x^2$. For all such rectangles, what are the dimensions of the one with the largest area?

13. If the position S of a particle moving along an s-axis is given as a function of the time t by $S(t) = 2t^3 - 9t^2 + 12t$ for $t > 0$,

    (a) find the velocity, $v(t)$ and acceleration, $a(t)$ of the particle;

    (b) find the average velocity, v_{av} of the particle over the time interval $t_1 = 1$ and $t_2 = 2$.

    (c) find all time intervals when the particle moves in the positive direction and when it moves in the negative direction. When is it stopped?

    (d) find all time intervals for $t > 0$ when the particle is speeding up and when it is slowing down.