Solution

Name: ______________________ ID Number: ____________

UNIVERSITY OF CALGARY
DEPARTMENT OF MATHEMATICS AND STATISTICS

MATHEMATICS 253 — L02 FALL 2004

QUIZ #4a [04-11-17(Wed)]

Attempt all problem. Each problem: 10 marks. Total: 30 marks (best 3 out of 4 problems).

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1. Find the area of the surface obtained by rotating about the x-axis the curve

$$x = 1 + 2y^2. \qquad 1 \le y \le 2$$

Surface area $= 2\pi \int_1^2 y\sqrt{1+x'^2}\,dy$ or $2\pi\int_3^9 y\sqrt{1+y'^2}\,dx$

$x' = 4y$
$x'^2 = 16y^2$

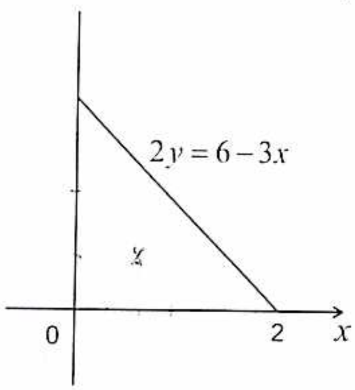
$= 2\pi\int_1^2 y\sqrt{1+16y^2}\,dy$

$= \frac{2\pi}{32}\int_1^2 32y(1+16y^2)^{\frac{1}{2}}\,dy$

$= \frac{\pi}{16}\ \frac{2}{3}(1+16y^2)^{\frac{3}{2}}\Big]_1^2$

$= \frac{\pi}{24}\left[(65)^{3/2} - (17)^{3/2}\right].$

2. Find the centre of mass $(\bar{x}, \bar{y})$ of the lamina shaped as given, assuming its density is 1.



$W = \int_0^2 y\,dx \qquad y = 3 - \frac{3}{2}x$

$= \int_0^2 3 - \frac{3x}{2}\,dx = 3x - \frac{3x^2}{4}\Big]_0^2$

$= 6 - 3 = 3.$

$\bar{x} = \dfrac{\int_0^2 xy\,dx}{W} = \frac{1}{3}\int_0^2 x(3 - \frac{3}{2}x)\,dx$

$= \frac{1}{3}\left(\frac{3x^2}{2} - \frac{3x^3}{2\cdot 3}\right)\Big]_0^2$

$= \frac{1}{3}(6 - 4) = \frac{2}{3}.$

$\therefore (\bar{x}, \bar{y}) = (\frac{2}{3}, 1)$

$\bar{y} = \dfrac{\frac{1}{2}\int_0^2 y^2\,dx}{3} = \frac{1}{6}\int_0^2 (3 - \frac{3x}{2})^2\,dx$

$= \frac{1}{6}\,\frac{1}{3\cdot(-\frac{3}{2})}\left(3 - \frac{3x}{2}\right)^3\Big]_0^2 = -\frac{1}{27}(-3)^3 = 1$

3. Solve $\frac{dy}{dx} = 3xy + x, \quad y(0) = 1$

$$\frac{dy}{dx} = x(3y+1) \qquad \text{Separable.}$$

$$\frac{dy}{3y+1} = x\,dx$$

Integrating $\displaystyle\int \frac{dy}{3y+1} = \int x\,dx$

$$\frac{1}{3} \ln \frac{3y+1}{c} = \frac{x^2}{2}.$$

$$\ln \frac{3y+1}{c} = \frac{3x^2}{2}$$

$$\therefore 3y+1 = c\,e^{\frac{3x^2}{2}}.$$

At $x=0$, $3+1 = c \cdot 1 \quad \therefore c = 4. \quad \therefore 3y+1 = e^{\frac{3x^2}{2}}.$

4. Find the orthogonal trajectories of the family of curves $y = ce^{-x}$, where c is an arbitrary constant.

$$y' = -ce^{-x} = -y$$

O.T. $\displaystyle\frac{dy}{dx} = \frac{-1}{-y} = \frac{1}{y}$

$$y\,dy = dx$$

$$\frac{y^2}{2} = x + c$$

Solution

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MATHEMATICS 253 — L02 FALL 2004

QUIZ #4b [04-11-18(Thur)]

Attempt all problem. Each problem: 10 marks. Total: 30 marks (best 3 out of 4 problems).

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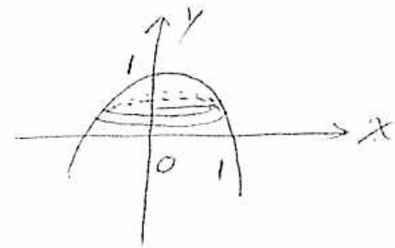
1. Find the area of the surface obtained by rotating about the y-axis the curve

$$y = 1 - x^2, \qquad 0 \le x \le 1$$



Surface area $= 2\pi \int_0^1 x\sqrt{1+y'^2}\,dx$

$= 2\pi \int_0^1 x\sqrt{1+4x^2}\,dx$

$y' = -2x.$

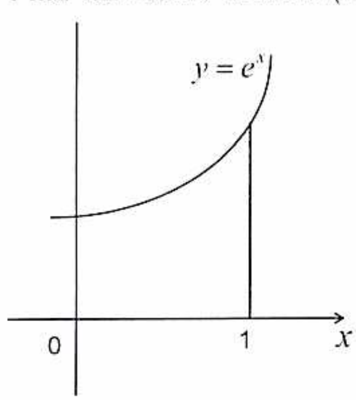
$y'^2 = 4x^2.$

$= \frac{2\pi}{8}\int_0^1 8x(1+4x^2)^{1/2}\,dx$

$= \frac{\pi}{4} \cdot \frac{2}{3}(1+4x^2)^{3/2}\Big]_0^1$

$= \frac{\pi}{6}(5^{3/2} - 1).$

2. Find the centre of mass $(\bar{x}, \bar{y})$ of the lamina shaped as given, assuming its density is 1.



$W = \int_0^1 y\,dx = \int_0^1 e^x\,dx = e^x\Big]_0^1$

$= e - 1.$

$\bar{x} = \dfrac{\int_0^1 xy\,dx}{W} = \dfrac{\int_0^1 xe^x\,dx}{e-1}$

$= \dfrac{xe^x - e^x\Big]_0^1}{e-1} = \dfrac{e^1 - e^1 + 1}{e-1}$

$= \dfrac{1}{e-1}.$

$\bar{y} = \frac{1}{2}\int_0^1 y^2\,dx \Big/ W$

$= \frac{1}{2}\int_0^1 e^{2x}\,dx = \frac{1}{2}\frac{1}{2}e^{2x}\Big]_0^1 = \underline{\frac{1}{4}(e^2-1)} = \frac{1}{4}(e+1)$

3. Solve $\frac{dy}{dx} = 2xy - x, \quad y(0) = 2$

$$\frac{dy}{dx} = x(2y-1) \qquad \text{Separable.}$$

$$\int \frac{dy}{2y-1} = \int x\,dx$$

$$\frac{1}{2} \ln \frac{2y-1}{c} = \frac{x^2}{2}$$

$$\frac{2y-1}{c} = e^{x^2}$$

$$2y-1 = ce^{x^2}$$

At $x=0$, $\quad 3 = c \cdot e^0 \quad \therefore c_3 = 3 \quad \therefore 2y-1 = 3e^{x^2}.$

4. Find the orthogonal trajectories of the family of curves $y^2 - x^2 = c$, where c is an arbitrary constant.

$$2y\frac{dy}{dx} - 2x = 0$$

$$\therefore \frac{dy}{dx} = \frac{2x}{2y} = \frac{x}{y}.$$

O.T. is given by

$$\frac{dy}{dx} = \frac{-1}{\frac{x}{y}} = \frac{-y}{x}.$$

$$\frac{dy}{y} = -\frac{dx}{x}$$

$$\ln\frac{y}{c} = -\ln x. = \ln x^{-1}$$

$$\therefore \frac{y}{c} = \frac{1}{x} \quad \text{or} \quad y = \frac{c}{x}. \quad \text{(hyperbolas)}$$