Summary: Solutions of $ay'' + by' + c = 0$

Roots of $ar^2 + br + c = 0$	General solution
r_1, r_2 real and distinct	$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$
$r_1 = r_2 = r$	$y = c_1 e^{rx} + c_2 x e^{rx}$
r_1, r_2 complex: $\alpha \pm i\beta$	$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$

17.1 Exercises

1–13 Solve the differential equation.

1. $y'' - 6y' + 8y = 0$
2. $y'' - 4y' + 8y = 0$
3. $y'' + 8y' + 41y = 0$
4. $2y'' - y' - y = 0$
5. $y'' - 2y' + y = 0$
6. $3y'' = 5y'$
7. $4y'' + y = 0$
8. $16y'' + 24y' + 9y = 0$
9. $4y'' + y' = 0$
10. $9y'' + 4y = 0$
11. $\dfrac{d^2y}{dt^2} - 2\dfrac{dy}{dt} - y = 0$
12. $\dfrac{d^2y}{dt^2} - 6\dfrac{dy}{dt} + 4y = 0$
13. $\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + y = 0$

14–16 Graph the two basic solutions of the differential equation and several other solutions. What features do the solutions have in common?

14. $6\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 2y = 0$
15. $\dfrac{d^2y}{dx^2} - 8\dfrac{dy}{dx} + 16y = 0$
16. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 5y = 0$

17–24 Solve the initial-value problem.

17. $2y'' + 5y' + 3y = 0$, $y(0) = 3$, $y'(0) = -4$
18. $y'' + 3y = 0$, $y(0) = 1$, $y'(0) = 3$
19. $4y'' - 4y' + y = 0$, $y(0) = 1$, $y'(0) = -1.5$
20. $2y'' + 5y' - 3y = 0$, $y(0) = 1$, $y'(0) = 4$
21. $y'' + 16y = 0$, $y(\pi/4) = -3$, $y'(\pi/4) = 4$
22. $y'' - 2y' + 5y = 0$, $y(\pi) = 0$, $y'(\pi) = 2$
23. $y'' + 2y' + 2y = 0$, $y(0) = 2$, $y'(0) = 1$
24. $y'' + 12y' + 36y = 0$, $y(1) = 0$, $y'(1) = 1$

25–32 Solve the boundary-value problem, if possible.

25. $4y'' + y = 0$, $y(0) = 3$, $y(\pi) = -4$
26. $y'' + 2y' = 0$, $y(0) = 1$, $y(1) = 2$
27. $y'' - 3y' + 2y = 0$, $y(0) = 1$, $y(3) = 0$
28. $y'' + 100y = 0$, $y(0) = 2$, $y(\pi) = 5$
29. $y'' - 6y' + 25y = 0$, $y(0) = 1$, $y(\pi) = 2$
30. $y'' - 6y' + 9y = 0$, $y(0) = 1$, $y(1) = 0$
31. $y'' + 4y' + 13y = 0$, $y(0) = 2$, $y(\pi/2) = 1$
32. $9y'' - 18y' + 10y = 0$, $y(0) = 0$, $y(\pi) = 1$

33. Let L be a nonzero real number.
    (a) Show that the boundary-value problem $y'' + \lambda y = 0$, $y(0) = 0$, $y(L) = 0$ has only the trivial solution $y = 0$ for the cases $\lambda = 0$ and $\lambda < 0$.
    (b) For the case $\lambda > 0$, find the values of λ for which this problem has a nontrivial solution and give the corresponding solution.
34. If a, b, and c are all positive constants and $y(x)$ is a solution of the differential equation $ay'' + by' + cy = 0$, show that $\lim_{x\to\infty} y(x) = 0$.

17.2 Nonhomogeneous Linear Equations

In this section we learn how to solve second-order nonhomogeneous linear differential equations with constant coefficients, that is, equations of the form

$$ay'' + by' + cy = G(x) \quad \boxed{1}$$

Then $$y_p'' = u_1' \cos x - u_2' \sin x - u_1 \sin x - u_2 \cos x$$

For y_p to be a solution we must have

$$\boxed{11} \qquad y_p'' + y_p = u_1' \cos x - u_2' \sin x = \tan x$$

Solving Equations 10 and 11, we get

$$u_1'(\sin^2 x + \cos^2 x) = \cos x \tan x$$

$$u_1' = \sin x \qquad u_1(x) = -\cos x$$

(We seek a particular solution, so we don't need a constant of integration here.) Then, from Equation 10, we obtain

$$u_2' = -\frac{\sin x}{\cos x} u_1' = -\frac{\sin^2 x}{\cos x} = \frac{\cos^2 x - 1}{\cos x} = \cos x - \sec x$$

||| Figure 5 shows four solutions of the differential equation in Example 7.

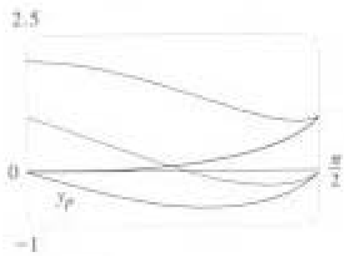

FIGURE 5

So $$u_2(x) = \sin x - \ln(\sec x + \tan x)$$

(Note that $\sec x + \tan x > 0$ for $0 < x < \pi/2$.) Therefore

$$\begin{aligned} y_p(x) &= -\cos x \sin x + [\sin x - \ln(\sec x + \tan x)] \cos x \\ &= -\cos x \ln(\sec x + \tan x) \end{aligned}$$

and the general solution is

$$y(x) = c_1 \sin x + c_2 \cos x - \cos x \ln(\sec x + \tan x)$$

||| 17.2 Exercises

1–10 ||| Solve the differential equation or initial-value problem using the method of undetermined coefficients.

1. $y'' + 3y' + 2y = x^2$

2. $y'' + 9y = e^{3x}$

3. $y'' - 2y' = \sin 4x$

4. $y'' + 6y' + 9y = 1 + x$

5. $y'' - 4y' + 5y = e^{-x}$

6. $y'' + 2y' + y = xe^{-x}$

7. $y'' + y = e^x + x^3, \quad y(0) = 2, \quad y'(0) = 0$

8. $y'' - 4y = e^x \cos x, \quad y(0) = 1, \quad y'(0) = 2$

9. $y'' - y' = xe^x, \quad y(0) = 2, \quad y'(0) = 1$

10. $y'' + y' - 2y = x + \sin 2x, \quad y(0) = 1, \quad y'(0) = 0$

11–12 ||| Graph the particular solution and several other solutions. What characteristics do these solutions have in common?

11. $4y'' + 5y' + y = e^x$

12. $2y'' + 3y' + y = 1 + \cos 2x$

13–18 ||| Write a trial solution for the method of undetermined coefficients. Do not determine the coefficients.

13. $y'' + 9y = e^{2x} + x^2 \sin x$

14. $y'' + 9y' = xe^{-x} \cos \pi x$

15. $y'' + 9y' = 1 + xe^{9x}$

16. $y'' + 3y' - 4y = (x^3 + x)e^x$

17. $y'' + 2y' + 10y = x^2e^{-x} \cos 3x$

18. $y'' + 4y = e^{3x} + x \sin 2x$

19–22 ||| Solve the differential equation using (a) undetermined coefficients and (b) variation of parameters.

19. $y'' + 4y = x$

20. $y'' - 3y' + 2y = \sin x$

21. $y'' - 2y' + y = e^{2x}$

22. $y'' - y' = e^x$

23–28 Solve the differential equation using the method of variation of parameters.

23. $y'' + y = \sec x$, $0 < x < \pi/2$

24. $y'' + y = \cot x$, $0 < x < \pi/2$

25. $y'' - 3y' + 2y = \dfrac{1}{1 + e^{-x}}$

26. $y'' + 3y' + 2y = \sin(e^x)$

27. $y'' - y = \dfrac{1}{x}$

28. $y'' + 4y' + 4y = \dfrac{e^{-2x}}{x^3}$

17.3 Applications of Second-Order Differential Equations

Second-order linear differential equations have a variety of applications in science and engineering. In this section we explore two of them: the vibration of springs and electric circuits.

Vibrating Springs

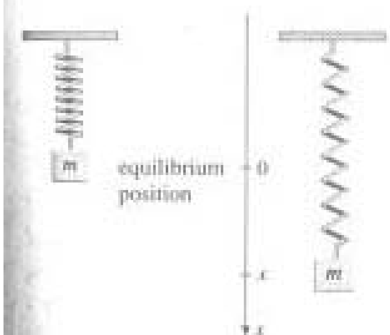

FIGURE 1

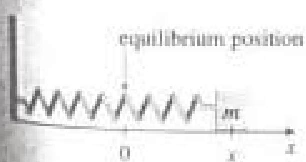

FIGURE 2

We consider the motion of an object with mass m at the end of a spring that is either vertical (as in Figure 1) or horizontal on a level surface (as in Figure 2).

In Section 6.4 we discussed Hooke's Law, which says that if the spring is stretched (or compressed) x units from its natural length, then it exerts a force that is proportional to x:

$$\text{restoring force} = -kx$$

where k is a positive constant (called the **spring constant**). If we ignore any external resisting forces (due to air resistance or friction) then, by Newton's Second Law (force equals mass times acceleration), we have

$$\boxed{1} \qquad m\frac{d^2x}{dt^2} = -kx \qquad \text{or} \qquad m\frac{d^2x}{dt^2} + kx = 0$$

This is a second-order linear differential equation. Its auxiliary equation is $mr^2 + k = 0$ with roots $r = \pm\omega i$, where $\omega = \sqrt{k/m}$. Thus, the general solution is

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t$$

which can also be written as

$$x(t) = A\cos(\omega t + \delta)$$

where

$$\omega = \sqrt{k/m} \quad \text{(frequency)}$$

$$A = \sqrt{c_1^2 + c_2^2} \quad \text{(amplitude)}$$

$$\cos\delta = \frac{c_1}{A} \qquad \sin\delta = -\frac{c_2}{A} \qquad (\delta \text{ is the phase angle})$$

(See Exercise 17.) This type of motion is called **simple harmonic motion**.