

Math 253 (L03) -1
Mid Term Review Solution

1. (a) $f(x) = \frac{1-x}{3x-2}$. Its inverse is:

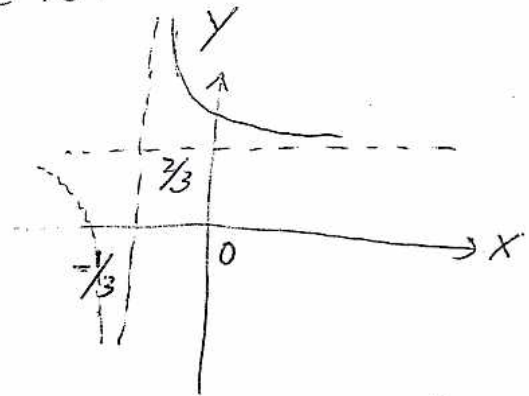
$$x = \frac{1-y}{3y-2}$$

$$(3y-2)x = 1-y$$

$$3yx - 2x + y = 1$$

$$y(3x+1) = 1+2x$$

$$y = \frac{1+2x}{3x+1}$$



Range: $y \neq \frac{2}{3}$.

Domain: $x \neq \frac{1}{3}$

(b). $f(x) = \sin(3x+2)$. Its inverse is:

$$x = \sin(3y+2)$$

$$\therefore \sin^{-1} x = 3y+2$$

$$\therefore y = \frac{1}{3}(\sin^{-1} x - 2)$$

Domain: $-1 \leq x \leq 1$

Range: $\frac{1}{3}(\frac{-\pi}{2} - 2) \leq y \leq \frac{1}{3}(\frac{\pi}{2} - 2)$

2. (a) $\int \sin^{-1} 2x \, dx$ (by parts)

$$= x \sin^{-1} 2x - \int x \cdot \frac{2}{\sqrt{1-4x^2}} \, dx$$

$$= \quad " \quad + \frac{1}{4} \int \frac{-8x}{\sqrt{1-4x^2}} \, dx$$

$$= \quad " \quad + \frac{1}{4} \cdot \frac{1}{\frac{1}{2}} (1-4x^2)^{\frac{1}{2}} + C$$

$$= \quad " \quad + \frac{1}{2} (1-4x^2)^{\frac{1}{2}} + C$$

Use $\int f' f^n \, dx = \frac{f^{n+1}}{n+1}, n \neq -1$

(Check answer by diff.)

2.(b) $\int_0^{\pi/4} x \sin 2x \, dx$ (by parts)

$$= -\frac{x}{2} \cos 2x \Big|_0^{\pi/4} + \frac{1}{2} \int_0^{\pi/4} \cos 2x \, dx$$

$$= -\frac{x}{2} \cos 2x + \frac{1}{4} \sin 2x \Big|_0^{\pi/4}$$

$$= -\frac{\pi}{8} \cos \frac{\pi}{2} + \frac{1}{4} \sin \frac{\pi}{2} - 0 - 0 = \frac{1}{4} \cdot 1 = \frac{1}{4}$$

(c) $\int_1^2 x (\ln x)^2 \, dx$ (by parts)

$$= \left[\frac{x^2}{2} (\ln x)^2 \right]_1^2 - \int_1^2 x^2 \cdot 2 \ln x \cdot \frac{1}{x} \, dx$$

$$= \left[\frac{x^2}{2} (\ln x)^2 \right]_1^2 - \int_1^2 x \ln x \, dx$$

(by parts again)

$$= \left[\frac{x^2}{2} (\ln x)^2 - \left(\frac{x^2}{2} \ln x - \frac{x^2}{4} \right) \right]_1^2$$

$\ln 1 = 0$.

$$= 2(\ln 2)^2 - 2 \ln 2 + 2 - \frac{1}{4}$$

$$= 2(\ln 2)^2 - 2 \ln 2 + \frac{3}{4}$$

(d) $\int \frac{x}{2x^2 - x + 2} \, dx$

Since $2x^2 - x + 2$ cannot be factorised, we complete the square:

$$= \int \frac{x}{2(x - \frac{1}{4})^2 + \frac{15}{8}} \, dx$$

Then let $u = x - \frac{1}{4}$
 $du = dx$

$$= \int \frac{u + \frac{1}{4}}{2u^2 + \frac{15}{8}} \, du$$

$$2(x^2 - \frac{x}{2}) + 2$$

$$= 2 \left[(x - \frac{1}{4})^2 - \frac{1}{4^2} \right] + 2$$

$$= 2(x - \frac{1}{4})^2 - \frac{1}{8} + 2$$

$$= 2(x - \frac{1}{4})^2 + \frac{15}{8}$$

$$= \int \frac{u}{2u^2 + \frac{15}{8}} \, du + \frac{1}{4} \int \frac{1}{2u^2 + \frac{15}{8}} \, du$$

$$= \frac{1}{4} \ln(2u^2 + \frac{15}{8}) + \frac{1}{4} \cdot \frac{1}{\sqrt{15/8}} \tan^{-1} \frac{\sqrt{2}u}{\sqrt{15/8}} + C$$

$$= \frac{1}{4} \ln(2u^2 + \frac{15}{8}) + \frac{1}{2\sqrt{15}} \tan^{-1} \frac{4u}{\sqrt{15}} + C,$$

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where $u = x - \frac{1}{4}$.

(e) $I = \int \frac{x+1}{2x^2-x-1} dx = \int \frac{x+1}{(2x+1)(x-1)} dx$ (partial fractions)

$$\frac{x+1}{(2x+1)(x-1)} = \frac{A}{2x+1} + \frac{B}{x-1}$$

$$x+1 = A(x-1) + B(2x+1)$$

Set:

$$\begin{array}{l} x = -\frac{1}{2} \quad \frac{1}{2} = A(-\frac{3}{2}) \quad \therefore A = -\frac{1}{3} \\ x = 1 \quad 2 = B \cdot 3 \quad \therefore B = \frac{2}{3} \end{array}$$

$$\therefore I = \int \frac{-\frac{1}{3}}{2x+1} + \frac{\frac{2}{3}}{x-1} dx$$

$$= -\frac{1}{3 \cdot 2} \ln|2x+1| + \frac{2}{3} \ln|x-1| + C$$

(f) $\int \frac{x^2}{\sqrt{4x^2+9}} dx$

Want: $4x^2 = 9 \tan^2 \theta$

ie $2x = 3 \tan \theta$

or $x = \frac{3}{2} \tan \theta$

$dx = \frac{3}{2} \sec^2 \theta d\theta$

$$= \int \frac{\frac{9}{4} \tan^2 \theta \cdot \frac{3}{2} \sec^2 \theta d\theta}{\sqrt{9 \tan^2 \theta + 9}}$$

$$= \int \frac{\frac{9}{4} \tan^2 \theta \cdot \frac{3}{2} \sec^2 \theta d\theta}{3 \sec \theta}$$

$$= \frac{9}{8} \int \tan^2 \theta \sec \theta d\theta$$

$$= \frac{9}{8} \int (\sec^2 \theta - 1) \sec \theta d\theta$$

$$= \frac{9}{8} \int \sec^3 \theta - \sec \theta d\theta$$

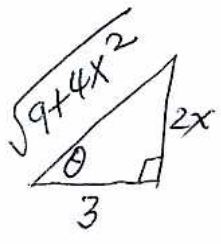
(or you can try $\int \frac{\sin^2 \theta}{\cos^2 \theta} \cdot \frac{1}{\cos \theta} d\theta$)

$$= \frac{9}{8} \int \frac{1}{2} [\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta| - \ln |\sec \theta + \tan \theta|] + C$$

$I = \int \sec^3 \theta d\theta = \int \sec \theta \sec^2 \theta d\theta$
(by parts)
 $= \sec \theta \tan \theta - \int \sec \theta \tan^2 \theta d\theta$

$$= \frac{9}{8} \cdot \frac{1}{2} (\sec \theta \tan \theta - \ln |\sec \theta + \tan \theta|) + C$$

$= \sec \theta \tan \theta - \int \sec \theta (\sec^2 \theta - 1) d\theta$
 $= \sec \theta \tan \theta - I + \int \sec \theta d\theta$



$$= \frac{9}{16} \left(\frac{2x\sqrt{9+4x^2}}{9} - \ln \left| \frac{\sqrt{9+4x^2} + 2x}{3} \right| \right) + C$$

$\therefore 2I = \dots + \ln |\sec \theta + \tan \theta|$
 $I = \frac{1}{2} [\dots]$

etc.

(g) $\int_0^1 x^2 e^{-2x} dx$ (by parts)

$$= x^2 \frac{e^{-2x}}{-2} + \frac{1}{2} \int 2x e^{-2x} dx$$

$$= -\frac{x^2 e^{-2x}}{2} + \left[\frac{x e^{-2x}}{-2} + \frac{1}{2} \int e^{-2x} dx \right]$$

$$= -e^{-2x} \left(\frac{x^2}{2} + \frac{x}{2} + \frac{1}{4} e^{-2x} \right) + C$$

3(a) Improper integral (at $x=0$)

$$\lim_{\delta \rightarrow 0^+} \int_{\delta}^2 \frac{dx}{x^{1/2}}$$

$$= \left[2x^{1/2} \right]_{\delta}^2$$

$$= 2(2)^{1/2} - 2(\delta)^{1/2}$$

$$= 2\sqrt{2}$$

$$(b) \lim_{N \rightarrow \infty} \int_1^N \ln x \, dx$$

$$= \int_1^N x \ln x \Big|_1^N - \int_1^N x \cdot \frac{1}{x} \, dx$$

$$= \int_1^N x \ln x - x \Big|_1^N$$

$$= N \ln N - N - (-1)$$

$$= \infty. \quad \text{Now, evaluate}$$

$$\lim_{N \rightarrow \infty} N \ln N = \infty.$$

$$(c) \lim_{N \rightarrow \infty} \int_1^N \frac{\ln x}{x^2} \, dx \quad (\text{by parts})$$

$$= \int_1^N -\frac{\ln x}{x} \Big|_1^N + \int_1^N \frac{1}{x} \cdot \frac{1}{x} \, dx = \int_1^N \frac{1}{x^2} \, dx$$

$$= \int_1^N -\frac{\ln x}{x} - \frac{1}{x} \Big|_1^N$$

$$= -\frac{\ln N}{N} - \frac{1}{N} + 1$$

$$= 1$$

now, evaluate (l'Hopital's rule):

$$\lim_{N \rightarrow \infty} \frac{\ln N}{N} = \lim_{N \rightarrow \infty} \frac{1/N}{1}$$

$$= \frac{1}{\infty} = 0$$

$$(d) \lim_{\delta \rightarrow 0^+} \int_{\delta}^1 x^2 \ln x \, dx$$

$$= \int_{\delta}^1 \frac{x^3 \ln x}{3} - \int_{\delta}^1 \frac{x^3}{3} \cdot \frac{1}{x} \, dx$$

$$= \int_{\delta}^1 \frac{x^3 \ln x}{3} - \frac{x^3}{9} \Big|_{\delta}^1$$

$$= \lim_{\delta \rightarrow 0^+} \left(-\frac{1}{9} - \frac{\delta^3 \ln \delta}{3} + \frac{\delta^3}{9} \right) \quad -6$$

$$= -\frac{1}{9}$$

now, evaluate

$$\lim_{\delta \rightarrow 0^+} \delta^3 \ln \delta = \lim_{\delta \rightarrow 0^+} \frac{\ln \delta}{\frac{1}{\delta^3}} \quad \left(\frac{-\infty}{\infty} \right)$$

$$= \lim_{\delta \rightarrow 0^+} \frac{\frac{1}{\delta}}{-\frac{3}{\delta^4}}$$

$$= \lim_{\delta \rightarrow 0^+} \frac{-\delta^3}{3} = 0$$

$$4.(a) F(x) = \int_{-x}^{2x} \sin 6x \, dx$$

$$\begin{aligned} \therefore F' &= 2 \cdot \sin 12x + 1 \cdot \sin(-6x) \\ &= 2 \sin 12x - \sin 6x \end{aligned}$$

$$\therefore F'' = 24 \cos 12x - 6 \cos 6x$$

$$(\sin -6x = -\sin 6x)$$

$$(b) F(x) = \int_{2x}^{x^2} \ln(x+1) \, dx$$

$$\therefore F' = 2x \ln(x^2+1) - 2 \ln(2x+1)$$

$$\therefore F'' = 2 \ln(x^2+1) + \frac{2x \cdot 2x}{x^2+1} - \frac{2 \cdot 2}{2x+1}$$

$$= 2 \ln(x^2+1) + \frac{4x^2}{x^2+1} - \frac{4}{2x+1}$$

$$5.(a) \quad y = \frac{1}{2}(e^x + e^{-x})$$

$$y' = \frac{1}{2}(e^x - e^{-x})$$

$$y'^2 = \frac{1}{4}(e^{2x} - 2 + e^{-2x})$$

$$1 + y'^2 = \frac{1}{4}(e^{2x} + 2 + e^{-2x}) \\ = \frac{1}{4}(e^x + e^{-x})^2$$

$$\therefore \text{Arc length} = \frac{1}{2} \int_0^1 e^x + e^{-x} dx$$

$$= \frac{1}{2} [e^x - e^{-x}]_0^1$$

$$= \frac{1}{2}(e - e^{-1} - (1 - 1))$$

$$= \frac{1}{2}(e - e^{-1}).$$

$$(b) \quad x = \frac{y^4}{8} + \frac{1}{4y^2}, \quad y = 1 \text{ to } y = 4.$$

$$x' = \frac{y^3}{2} - \frac{1}{y^3} = \frac{1}{2} \left(y^3 - \frac{1}{y^3} \right)$$

$$x'^2 = \frac{1}{4} \left(y^3 - \frac{1}{y^3} \right)^2 = \frac{1}{4} \left(y^6 - 2 + \frac{1}{16y^6} \right)$$

$$1 + x'^2 = \frac{1}{4} \left(y^6 + 2 + \frac{1}{y^6} \right) = \frac{1}{4} \left(y^3 + \frac{1}{y^3} \right)^2$$

$$\therefore \text{Arc length} = \int_1^4 \sqrt{1 + x'^2} dy$$

$$= \frac{1}{2} \int_1^4 \left(y^3 + \frac{1}{y^3} \right) dy = \frac{1}{2} \left[\frac{y^4}{4} - \frac{1}{2y^2} \right]_1^4$$

$$= \frac{1}{2} \left(64 - \frac{1}{32} - \left(\frac{1}{4} - \frac{1}{2} \right) \right) = 32 - \frac{7}{64}.$$

$$5(c) \quad \text{Arc length} = \int_0^{\pi/4} \sqrt{x'^2 + y'^2} dt$$

$$x' = -2\sin 2t \quad y' = 2\cos 2t.$$

$$x'^2 + y'^2 = 4(\sin^2 2t + \cos^2 2t) = 4.$$

$$\therefore \text{Arc length} = \int_0^{\pi/4} 2 dt = 2t \Big|_0^{\pi/4} = \frac{\pi}{2}.$$