Name: SOLUTION KEY

UNIVERSITY OF CALGARY
DEPARTMENT OF MATHEMATICS AND STATISTICS

MATHEMATICS 253 — L60 SUMMER 2006

MIDTERM EXAMINATION

Instructor: F. Fodor Date: July 26, 2006, 11:00–13:00

Attempt all problems. Explain your answers clearly. Read questions carefully. Show all relevant work. A distribution of marks can be located on the last page of the examination. You may use a non-graphing calculator and a formula sheet.

Marks

[5]

1. Solve the equation.

$\ln(4x) - 3\ln(x^2) = \ln 2$

$$\ln(4x) - \ln(x^2)^3 = \ln 2$$

$$\ln\left(\frac{4x}{x^6}\right) = \ln 2$$

$$\frac{4}{x^5} = 2$$

$$x = \sqrt[5]{2}$$

[5]

2. Find the exact value of

$\tan(\sin^{-1}(-\frac{1}{2}))$.

$$\tan\left(\sin^{-1}\left(-\tfrac{1}{2}\right)\right) = \frac{\sin\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)}{\cos\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)} =$$

$$= \frac{-\frac{1}{2}}{\sqrt{1-\left(-\frac{1}{2}\right)^2}} = \frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

[5] 3. Evaluate the integral.

$\int \frac{x^2}{1+x^6} dx$ $= \int \frac{x^2}{1+(x^3)^2} dx = \frac{1}{3} \int \frac{du}{1+u^2} = \frac{1}{3} \tan^{-1} u + C =$

$u = x^3, \ du = 3x^2 dx$

$= \underline{\underline{\frac{1}{3} \tan^{-1}(x^3) + C}}$

[5] 4. Evaluate the definite integral.

$\int_1^3 \frac{dx}{\sqrt{x}(x+1)}$ $= \int_1^{\sqrt{3}} \frac{2\,du}{u^2+1} = 2 \tan^{-1} u \Big]_1^{\sqrt{3}} = 2\left(\tan^{-1}(\sqrt{3}) - \tan^{-1}(1)\right)$

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}} dx$

$= 2\left(\frac{\pi}{3}\right) - 2\left(\frac{\pi}{4}\right) = \underline{\frac{\pi}{6}}$

$u(1) = 1$

$u(3) = \sqrt{3}$

[10] 5. Evaluate the integral.

$\int \tan x \sec^3 x dx = \int \sec^2 x (\sec x \tan x) dx = \int u^2 du = \frac{u^3}{3} + C = \frac{\sec^3 x}{3} + C$

$u = \sec x$

$du = \sec x \tan x \, dx$

[10] 6. Evaluate the integral.

$\int \frac{1}{x^3\sqrt{5-x^2}} dx = \int \frac{\cancel{\sqrt{5}} \cancel{\cos\theta} d\theta}{(\sqrt{5}\sin\theta)^3 \cancel{\sqrt{5}} \cancel{\cos\theta}} = \frac{1}{5\sqrt{5}} \int \frac{1}{\sin^3\theta} d\theta =$

$x = \sqrt{5}\sin\theta$

$dx = \sqrt{5}\cos\theta d\theta$

$= \frac{1}{(\sqrt{5})^3} \int \csc^3\theta \, d\theta = \frac{1}{(\sqrt{5})^3}\left[-\frac{1}{2}\csc\theta\cot\theta + \frac{1}{2}\int \csc\theta \, d\theta\right] =$

USE FORMULA SHEET

$= \frac{1}{(\sqrt{5})^3}\left[-\frac{1}{2}\csc\theta\cot\theta + \frac{1}{2}\ln|\csc\theta - \cot\theta|\right] + C = (*)$

USE FORMULA SHEET

$\csc\theta = \frac{\sqrt{5}}{x}$, $\cot\theta = \frac{\sqrt{5}\sqrt{1-\left(\frac{x}{\sqrt{5}}\right)^2}}{x} = \frac{\sqrt{5-x^2}}{x}$

$(*) = \frac{1}{(\sqrt{5})^3}\left[-\frac{1}{2}\frac{\sqrt{5}\sqrt{5-x^2}}{x^2} + \frac{1}{2}\ln\left|\frac{\sqrt{5}}{x} - \frac{\sqrt{5-x^2}}{x}\right|\right] + C$

[10] 7. Evaluate the integral.

$\int \frac{5x-5}{3x^2-8x-3}dx = (*)$

$$3x^2 - 8x - 3 = (3x+1)(x-3)$$

$$\frac{5x-5}{(3x+1)(x-3)} = \frac{A}{3x+1} + \frac{B}{x-3} = \frac{A(x-3) + B(3x+1)}{(3x+1)(x-3)}$$

$$5x - 5 = A(x-3) + B(3x+1)$$

$$5x - 5 = (A+3B)x + B - 3A$$

$$\begin{cases} 5 = A + 3B \longrightarrow 5 = A + 3(3A-5) = 10A - 15 \to \boxed{A = 2} \\ -5 = B - 3A \longrightarrow B = 3A - 5 \longrightarrow \boxed{B = 1} \end{cases}$$

$$(*) = \int \frac{2}{3x+1}dx + \int \frac{1}{x-3}dx =$$

$$= \frac{2}{3}\ln|3x+1| + \ln|x-3| + C$$