THE UNIVERSITY OF CALGARY
DEPARTMENT OF MATHEMATICS AND STATISTICS
FINAL EXAMINATION
MATH 253 L(60) - SUMMER 2006

Saturday, August 19, 2006 12:00-3:00 pm (F. Fodor) Time: 3 hours

I.D. NUMBER	SURNAME	OTHER NAMES
	SOLUTION KEY	

STUDENT IDENTIFICATION

Each candidate must sign the Seating List confirming presence at the examination. All candidates for final examinations are required to place their University of Calgary student I.D. cards on their desks for the duration of the examination. (Students writing mid-term tests can also be asked to provide identity proof.) Students without an I.D. card who can produce an **acceptable** alternative I.D., e.g., one with a printed name and photograph, are allowed to write the examination.

A student without acceptable I.D. will be required to complete an Identification Form. The form indicates that there is no guarantee that the examination paper will be graded if any discrepancies in identification are discovered after verification with the student's file. **A student who refuses to produce identification or who refuses to complete and sign the Identification Form is not permitted to write the examination.**

EXAMINATION RULES

1. Students late in arriving will not normally be admitted after one-half hour of the examination time has passed.
2. No candidate will be permitted to leave the examination room until one-half hour has elapsed after the opening of the examination, nor during the last 15 minutes of the examination. All candidates remaining during the last 15 minutes of the examination period must remain at their desks until their papers have been collected by an invigilator.
3. All enquiries and requests must be addressed to supervisors only.
4. **Candidates are strictly cautioned against**:
   (a) speaking to other candidates or communicating with them under any circumstances whatsoever;
   (b) bringing into the examination room any textbook, notebook or memoranda not authorized by the examiner;
   (c) making use of calculators and/or portable computing machines not authorized by the instructor;
   (d) leaving answer papers exposed to view;
   (e) attempting to read other students' examination papers.

   The penalty for violation of these rules is suspension or expulsion or such other penalty as may be determined.
5. Candidates are requested to write on both sides of the page, unless the examiner has asked that the left half page be reserved for rough drafts or calculations.
6. Discarded matter is to be struck out and not removed by mutilation of the examination answer book.
7. Candidates are cautioned against writing in their answer book any matter extraneous to the actual answering of the question set.
8. The candidate is to write his/her name on each answer book as directed and is to number each book.
9. During the examination a candidate must report to a supervisor before leaving the examination room.
10. Candidates must stop writing when the signal is given. Answer books must be handed to the supervisor-in-charge promptly. Failure to comply with these regulations will be cause for rejection of an aswer paper.
11. If during the course of an examination a student becomes ill or receives word of domestic affliction, the student must report at once to the supervisor, hand in the unfinished paper and request that it be cancelled. If physical and/or emotional ill health is the cause, the student must report at once to a physician/counsellor so that subsequent application for a deferred examination is supported by a completed Physical/Counsellor Statement form. Students can consult professionals at University Health Services or Counselling and Student Development Centre during normal working hours or consult their physician/counsellor in the community. **Once an examination has been handed in for marking a student cannot request that the examination be cancelled for whatever reason. Such a request will be denied. Retroactive withdrawals will also not be considered.**

Question	Total Marks	Actual Marks
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total	100	

NOTE: A non-graphing calculator and a formula sheet *are* allowed.

1. [10 marks] Differentiate the function.

$$y = \pi^{x \tan x}$$

$\ln y = \ln\left(\pi^{x\tan x}\right) = x\tan x \cdot \ln \pi$

$\frac{1}{y} y' = \ln \pi \left(\tan x + x \sec^2 x\right)$

$y' = \pi^{x\tan x} \ln \pi \left(\tan x + x \sec^2 x\right)$

2. [10 marks] Evaluate the indefinite integral.

$$\int \cos^{-1}(2x)\, dx \quad = \quad (*)$$

$u = 2x$

$du = 2\,dx$

$(*) = \frac{1}{2} \int \cos^{-1} u \, du = \frac{1}{2} u \cos^{-1} u - \sqrt{1-u^2}\,\frac{1}{2} + C =$

$= x\cos^{-1}(2x) - \frac{1}{2}\sqrt{1-4x^2} + C$

3. [10 marks] Evaluate the indefinite integral.

$$\int \sin^5 x \cos^2 x \, dx = (*)$$

$u = \cos x$

$du = -\sin x \, dx$

$$(*) = \int \sin^4 x \cos^2 x \sin x \, dx = -\int (1-u^2)^2 u^2 \, du =$$

$$= -\int u^2 - 2u^4 + u^6 \, du = -\frac{u^3}{3} + \frac{2}{5}u^5 - \frac{u^7}{7} + C =$$

$$= -\frac{\cos^3 x}{3} + \frac{2}{5}\cos^5 x - \frac{\cos^7 x}{7} + C$$

4. [10 marks] Evaluate the indefinite integral.

$$\int \frac{2x^2+3}{x(x-1)^2} \, dx = \int \frac{3}{x} + \frac{-1}{x-1} + \frac{5}{(x-1)^2} \, dx$$

$$\frac{2x^2+3}{x(x-1)^2} = \frac{A}{x} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2} \qquad = 3\ln|x| - \ln|x-1| - \frac{5}{x-1} +$$

$A(x-1)^2 + Bx(x-1) + Cx = 2x^2 + 3$

$A + B = 2 \longrightarrow \boxed{B = -1}$

$-2A - B + C = 0 \quad \boxed{C = 5}$

$\boxed{A = 3}$

5. [10 marks] Evaluate the improper integral.

$$\int_{-\infty}^{+\infty} \frac{x}{(x^2+3)^2}\,dx =$$

$$= \int_{-\infty}^{0} \frac{x}{(x^2+3)^2}\,dx + \int_{0}^{+\infty} \frac{x}{(x^2+3)^2}\,dx = \lim_{a\to-\infty} \int_{a}^{0} \frac{x}{(x^2+3)^2}\,dx +$$

$$+ \lim_{b\to\infty} \int_{0}^{b} \frac{x}{(x^2+3)^2}\,dx =$$

$$= \lim_{a\to-\infty} \frac{1}{2}\frac{-1}{x^2+3}\Big]_a^0 + \lim_{b\to+\infty} \frac{1}{2}\frac{-1}{x^2+3}\Big]_0^b =$$

$$= \lim_{a\to-\infty} \frac{1}{2}\frac{+1}{x^2+3} + \lim_{b\to\infty} \frac{1}{2}\frac{-1}{x^2+3} = 0$$

6. [10 marks] Find the volume of the solid that results when the region enclosed by the given curves is revolved about the x-axis.

$$y = \frac{1}{\sqrt{4+x^2}},\ x = -2,\ x = 2,\ y = 0$$

$$V = \int_{-2}^{2} \pi\left(\frac{1}{\sqrt{4+x^2}}\right)^2 dx = \pi\int_{-2}^{2} \frac{1}{4+x^2}\,dx = \pi\int_{-2}^{2} \frac{\frac{1}{4}\cdot 1}{1+\frac{x^2}{4}}\,dx =$$

$u = \frac{x}{2}$, $du = \frac{1}{2}$

$$= \frac{1}{2}\pi\int_{-1}^{1} \frac{1}{1+u^2}\,du = \frac{1}{2}\pi \tan^{-1}u\Big]_{-1}^{1} =$$

$$= \frac{1}{2}\pi\left(\tan^{-1}(1) - \tan^{-1}(-1)\right) = \frac{1}{2}\pi\left(\frac{\pi}{4} - -\frac{\pi}{4}\right) =$$

$$= \frac{\pi^2}{4}$$

7. [10 marks] Find the exact arc length of the curve over the stated interval.

$$y = x^{2/3} \text{ from } x = 1 \text{ to } x = 8$$

$\frac{dy}{dx} = \frac{2}{3} x^{-1/3}$

$L = \int_1^8 \sqrt{1 + \frac{4}{9} x^{-4/3}}\, dx = \int_1^8 \frac{2}{3} x^{-1/3} \sqrt{\frac{9}{4} x^{2/3} + 1}\, dx$

$u = \frac{9}{4} x^{2/3} + 1$

$du = \frac{3}{2} x^{-1/3} dx$

$u(1) = \frac{9}{4} + 1 = \frac{13}{4}$

$u(8) = \frac{9}{4} \cdot 8^{2/3} + 1$

$= 9 + 1 = 10$

$= \frac{4}{9} \int_{13/4}^{10} \sqrt{u}\, du = \frac{4}{9} \left[\frac{u^{3/2}}{3/2} \right]_{13/4}^{10} =$

$= \frac{8}{27} \left(10^{3/2} - \left(\frac{13}{4}\right)^{3/2} \right)$

8. [10 marks] Use sigma notation to write the Taylor series about $x = x_0$ for the function.

$$f(x) = \frac{1}{x+2}, \quad x_0 = 3$$

$f'(x) = \frac{-1}{(x+2)^2}$

$f''(x) = \frac{2}{(x+2)^3}$

$f'''(x) = \frac{-6}{(x+2)^4}$

$\vdots$

$f^{(n)}(x) = \frac{(-1)^n n!}{(x+2)^{n+1}}$

$f'(3) = \frac{-1}{25}$

$f''(3) = \frac{2}{125}$

$f'''(3) = \frac{-6}{5^4}$

$\vdots$

$f^{(n)}(3) = \frac{(-1)^n n!}{5^{n+1}}$

Taylor series

$$\sum_{k=0}^{\infty} (-1)^k \frac{1}{5^{k+1}} (x-3)^k$$

9. [10 marks] Solve the differential equation.

$$\frac{dy}{dx} + y + \frac{1}{1-e^x} = 0$$

$$\frac{dy}{dx} + y = \frac{-1}{1-e^x}$$

$$\mu = e^{\int p(x)\,dx} = e^{\int 1\,dx} = e^x$$

$$\int \mu(x)\, q(x)\,dx = \int e^x \frac{1}{e^x - 1}\,dx = \ln|e^x - 1| + C$$

$$y = \frac{1}{\mu(x)} \int \mu(x)\, q(x)\,dx = e^{-x}\left(\ln|e^x - 1| + C\right) =$$

$$= \frac{\ln|e^x - 1|}{e^x} + C e^{-x}$$

10. [10 marks] Find the general solution of the differential equation.

$$y'' - 6y' + 9y = 0$$

Auxiliary equation:

$$m^2 - 6m + 9 = 0$$

$$(m-3)^2 = 0$$

$$\boxed{m = 3}$$

General solution: $y = C_1 e^{3x} + C_2 x e^{3x}$

End of Examination