Math 253 (L03)
Mid Term Review Solution

1. (a) $f(x)=\dfrac{1-x}{3x-2}$. Its inverse is:

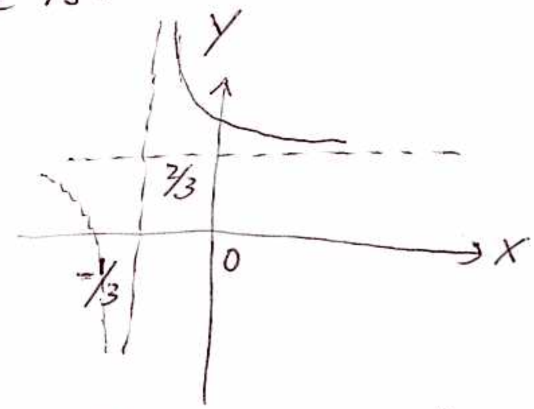
$$x=\frac{1-y}{3y-2}$$

$$(3y-2)x=1-y.$$

$$3yx-2x+y=1$$

$$y(3x+1)=1+2x$$

$$y=\frac{1+2x}{3x+1}.$$



Range: $y \neq \frac{2}{3}$.
Domain: $x \neq -\frac{1}{3}$

(b). $f(x)=\sin(3x+2)$. Its inverse is:

$$x=\sin(3y+2)$$

$$\therefore \sin^{-1}x=3y+2$$

$$\therefore y=\frac{1}{3}(\sin^{-1}x-2)$$

Domain: $-1\le x\le 1$
Range: $\frac{1}{3}(-\frac{\pi}{2}-2)\le y\le \frac{1}{3}(\frac{\pi}{2}-2)$

2. (a) $\int \sin^{-1}2x\,dx$ (by parts)

$$= x\sin^{-1}2x-\int x\cdot\frac{2}{\sqrt{1-4x^2}}\,dx$$

$$= \text{"}\quad +\frac{1}{4}\int\frac{-8x}{\sqrt{1-4x^2}}\,dx$$

use $\int f'f^n\,dx = \frac{f^{n+1}}{n+1}$, $n\neq -1$.

$$= \text{"}\quad +\frac{1}{4}\cdot\frac{1}{\frac{1}{2}}(1-4x^2)^{1/2}+C$$

$$= \text{"}\quad +\frac{1}{2}(1-4x^2)^{1/2}+C$$

(Check answer by diff.)

2.(b) $\int_0^{\pi/4} x \sin 2x \, dx$ (by parts)

$= -\frac{x}{2}\cos 2x\Big]_0^{\pi/4} + \frac{1}{2}\int_0^{\pi/4} \cos 2x \, dx$

$= -\frac{x}{2}\cos 2x + \frac{1}{4}\sin 2x\Big]_0^{\pi/4}$

$= -\frac{\pi}{8}\cos\frac{\pi}{2} + \frac{1}{4}\sin\frac{\pi}{2} - 0 - 0 = \frac{1}{4}\cdot 1 = \frac{1}{4}$

(c) $\int_1^2 x(\ln x)^2 \, dx$ (by parts)

$= \frac{x^2}{2}(\ln x)^2\Big]_1^2 - \frac{1}{2}\int_1^2 x^2 \cdot 2\ln x \cdot \frac{1}{x} \, dx$

$= \frac{x^2}{2}(\ln x)^2\Big]_1^2 - \int_1^2 x \ln x \, dx$ (by parts again)

$= \frac{x^2}{2}(\ln x)^2 - \left(\frac{x^2}{2}\ln x - \frac{x^2}{4}\right)\Big]_1^2$

$\ln 1 = 0$.

$= 2(\ln 2)^2 - 2\ln 2 + 2 - \frac{1}{4}$

$= 2(\ln 2)^2 - 2\ln 2 + 1\frac{3}{4}$

(d) $\int \frac{x}{2x^2 - x + 2} \, dx$

Since $2x^2 - x + 2$ cannot be factorised, we complete the square:

$2(x^2 - \frac{x}{2}) + 2$

$= 2\left[(x - \frac{1}{4})^2 - \frac{1}{4^2}\right] + 2$

$= 2(x - \frac{1}{4})^2 - \frac{1}{8} + 2$

$= 2(x - \frac{1}{4})^2 + \frac{15}{8}$

$= \int \frac{x}{2(x - \frac{1}{4})^2 + \frac{15}{8}} \, dx$

Then let $u = x - \frac{1}{4}$

$du = dx$

$= \int \frac{u + \frac{1}{4}}{2u^2 + \frac{15}{8}} \, du$

$= \int \frac{u}{2u^2 + \frac{15}{8}} + \frac{\frac{1}{4}}{2u^2 + \frac{15}{8}} \, du$.

$= \frac{1}{4}\ln\left(2u^2 + \frac{15}{8}\right) + \frac{1}{4}\frac{1}{\sqrt{2}\sqrt{15/8}}\tan^{-1}\frac{\sqrt{2}\,u}{\sqrt{15/8}} + C$

$$= \frac{1}{4} \ln\left(2u^2+\frac{15}{8}\right) + \frac{1}{2\sqrt{15}} \tan^{-1}\frac{4u}{\sqrt{15}} + C,$$

where $u = x - \frac{1}{4}$.

(e). $I = \int \frac{x+1}{2x^2-x-1}\,dx = \int \frac{x+1}{(2x+1)(x-1)}\,dx$ (partial fractions)

$$\frac{x+1}{(2x+1)(x-1)} = \frac{A}{2x+1} + \frac{B}{x-1}$$

$$x+1 = A(x-1) + B(2x+1)$$

Set:

$x = -\frac{1}{2}$ $\quad \frac{1}{2} = A\left(-\frac{3}{2}\right)$ $\quad \therefore A = -\frac{1}{3}.$

$x = 1$ $\quad 2 = B \cdot 3$ $\quad \therefore B = \frac{2}{3}.$

$$\therefore I = \int \frac{-\frac{1}{3}}{2x+1} + \frac{\frac{2}{3}}{x-1}\,dx$$

$$= -\frac{1}{3\cdot 2} \ln|2x+1| + \frac{2}{3}\ln|x-1| + C$$

(f) $\int \frac{x^2}{\sqrt{4x^2+9}}\,dx.$

Want: $4x^2 = 9\tan^2\theta.$

ie $2x = 3\tan\theta$

ie $x = \frac{3}{2}\tan\theta.$

$dx = \frac{3}{2}\sec^2\theta\,d\theta$

$$= \int \frac{\frac{9}{4}\tan^2\theta}{\sqrt{9\tan^2\theta+9}} \cdot \frac{3}{2}\sec^2\theta\,d\theta$$

$$= \int \frac{\frac{9}{4}\tan^2\theta \cdot \frac{3}{2}\sec^2\theta\,d\theta}{3\sec\theta}$$

$$= \frac{9}{8}\int \tan^2\theta \sec\theta\,d\theta.$$

$$= \frac{9}{8}\int (\sec^2\theta - 1)\sec\theta \, d\theta$$

(Or you can try $\int \frac{\sin^2\theta}{\cos^2\theta} \cdot \frac{1}{\cos\theta} \, d\theta$)

$$= \frac{9}{8}\int \sec^3\theta - \sec\theta \, d\theta$$

$$= \frac{9}{8}\left\{\frac{1}{2}\left[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right] - \ln|\sec\theta + \tan\theta|\right\} + C$$

$$I = \int \sec^3\theta \, d\theta = \int \sec\theta \sec^2\theta \, d\theta \quad \text{(by parts)}$$

$$= \sec\theta . \tan\theta - \int \sec\theta\tan\theta . \tan\theta \, d\theta$$

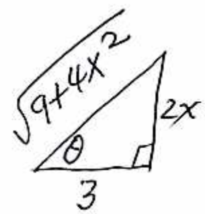
$$= \sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1) . d\theta$$

$$= \quad '' \quad - I + \int \sec\theta \, d\theta$$

$$\therefore 2I = \quad '' \quad + \ln|\sec\theta + \tan\theta|$$

$$I = \frac{1}{2}[\; - - - - \;]$$

$$= \frac{9}{8} \cdot \frac{1}{2}\left(\sec\theta\tan\theta - \ln|\sec\theta + \tan\theta|\right) + C$$



$$= \frac{9}{16}\left(\frac{2x\sqrt{9+4x^2}}{9} - \ln\left|\frac{\sqrt{9+4x^2}+2x}{3}\right|\right) + C$$

etc.

(g) $\int_0^1 x^2 e^{-2x} \, dx$ (by parts)

$$= x^2 \frac{e^{-2x}}{-2} + \frac{1}{\cancel{2}}\int \cancel{2}x e^{-2x} \, dx$$

$$= -\frac{x^2 e^{-2x}}{2} + \left[x\frac{e^{-2x}}{-2} + \frac{1}{2}\int e^{-2x} \, dx\right]$$

$$= -e^{-2x}\left(\frac{x^2}{2} + \frac{x}{2} + \frac{1}{4}e^{-2x}\right) + C.$$

3 (a) Improper integral (at $x=0$)

$$\lim_{\delta \to 0^+} \int_\delta^2 \frac{dx}{x^{1/2}}$$

$$= \quad '' \quad 2x^{1/2}\Big]_\delta^2$$

$$= \quad '' \quad 2(2)^{1/2} - 2(\delta)^{1/2}$$

$$= 2\sqrt{2}.$$

(b) $\lim_{N\to\infty} \int_1^N \ln x \, dx$

$= \text{"} \quad x \ln x \Big]_1^N - \int_1^N x \cdot \frac{1}{x} \, dx$

$= \text{"} \quad x \ln x - x \Big]_1^N$

$= \text{"} \quad N \ln N - N - (-1)$

$= \infty.$

Now, evaluate
$\lim_{N\to\infty} N \ln N = \infty.$

(c) $\lim_{N\to\infty} \int_1^N \frac{\ln x}{x^2} \, dx$ (by parts)

$= \text{"} \quad -\frac{\ln x}{x}\Big]_1^N + \int_1^N \frac{1}{x} \cdot \frac{1}{x} \, dx$ — $\int \frac{1}{x^2} \, dx$

$= \text{"} \quad -\frac{\ln x}{x} - \frac{1}{x} \Big]_1^N$

$= \text{"} \quad -\frac{\ln N}{N} - \frac{1}{N} + 1.$

$= 1$

Now, evaluate (l'Hôpital's rule):
$\lim_{N\to\infty} \frac{\ln N}{N} = \lim_{N\to\infty} \frac{\frac{1}{N}}{1}$
$= \frac{1}{\infty} = 0$

(d) $\lim_{\delta\to 0^+} \int_\delta^1 x^2 \ln x \, dx$

$= \text{"} \quad \frac{x^3}{3} \ln x - \int \frac{x^3}{3} \cdot \frac{1}{x} \, dx$

$= \text{"} \quad \frac{x^3}{3} \ln x - \frac{x^3}{9} \Big]_\delta^1$

$$= \lim_{\delta \to 0^+} -\frac{1}{9} - \frac{\delta^3}{3}\ln\delta + \frac{\delta^3}{9}$$

$$= -\frac{1}{9}.$$

now, evaluate

$$\lim_{\delta\to 0^+} \delta^3 \ln\delta = \lim_{\delta\to 0^+} \frac{\ln\delta}{\frac{1}{\delta^3}} \quad \left(\frac{-\infty}{\infty}\right)$$

$$= " \quad \frac{\frac{1}{\delta}}{\frac{-3}{\delta^4}}$$

$$= " \quad \frac{-\delta^3}{3} = 0$$

4.(a) $F(x) = \int_{-x}^{2x} \sin 6x \, dx$

$\therefore F' = 2 \cdot \sin 12x + 1 \cdot \sin(-6x)$ $\quad (\sin -6x = -\sin 6x)$

$= 2\sin 12x - \sin 6x.$

$\therefore F'' = 24\cos 12x - 6\cos 6x$

(b) $F(x) = \int_{2x}^{x^2} \ln(x+1)\, dx$

$\therefore F' = 2x\ln(x^2+1) - 2\ln(2x+1)$

$\therefore F'' = 2\ln(x^2+1) + \frac{2x.2x}{x^2+1} - \frac{2.2.}{2x+1}$

$= 2\ln(x^2+1) + \frac{4x^2}{x^2+1} - \frac{4}{2x+1}.$

5.(a) $y=\frac{1}{2}(e^{x}+e^{-x})$

$y'=\frac{1}{2}(e^{x}-e^{-x})$

$y'^2=\frac{1}{4}(e^{2x}-2+e^{-2x})$

$1+y'^2=\frac{1}{4}(e^{2x}+2+e^{-2x})$

$=\frac{1}{4}(e^{x}+e^{-x})^2$

$\therefore$ Arc length $=\frac{1}{2}\int_0^1 e^{x}+e^{-x}\,dx$

$=\frac{1}{2}\left[e^{x}-e^{-x}\right]_0^1$

$=\frac{1}{2}\left(e-e^{-1}-(1-1)\right)$

$=\frac{1}{2}(e-e^{-1})$

(b) $x=\frac{y^4}{8}+\frac{1}{4y^2}$, $y=1$ to $y=4$.

$x'=\frac{y^3}{2}-\frac{1}{y^3}=\frac{1}{2}\left(y^3-\frac{1}{y^3}\right)$

$x'^2=\frac{1}{4}\left(y^3-\frac{1}{y^3}\right)^2=\frac{1}{4}\left(y^6-2+\frac{1}{16y^6}\right)$.

$1+x'^2=\frac{1}{4}\left(y^6+2+\frac{1}{y^6}\right)=\frac{1}{4}\left(y^3+\frac{1}{y^3}\right)^2$

$\therefore$ Arc length $=\int_1^4\sqrt{1+x'^2}\,dy$

$=\frac{1}{2}\int_1^4 y^3+\frac{1}{y^3}\,dy=\frac{1}{2}\left[\frac{y^4}{4}-\frac{1}{2y^2}\right]_1^4$

$=\frac{1}{2}\left(64-\frac{1}{32}-\left(\frac{1}{4}-\frac{1}{2}\right)\right)=32-\frac{7}{64}$.

You can ignore this problem.

5(c) $y = \sqrt{9-4x^2}$ is an ellipse. ie $y^2 = 9-4x^2$

or $y^2 + 4x^2 = 9$

The arc length of an ellipse is obtained by re-writing it into:

or $\frac{y^2}{9} + \frac{4x^2}{9} = 1$

$$\begin{cases} 2x = 3\cos\theta \\ y = 3\sin\theta. \end{cases}$$

and then use:

$$\text{Arc length} = \int_{\theta_1}^{\theta_2} \sqrt{x'^2 + y'^2}\, d\theta \qquad \text{as in } 5(d)$$

$$= \int_{\theta_1}^{\theta_2} \sqrt{\tfrac{9}{4}\sin^2\theta + 9\cos^2\theta}\, d\theta \qquad x' = -\tfrac{3}{2}\sin\theta, \quad y' = 3\cos\theta$$

$$= \int_{\theta_1}^{\theta_2} \sqrt{\tfrac{9}{4} + \left(9 - \tfrac{9}{4}\right)\cos^2\theta}\, d\theta \qquad \sin^2\theta = 1 - \cos^2\theta.$$

Such "elliptic integral" can be obtained approximately.

5(d) $\text{Arc length} = \int_0^{\pi/4} \sqrt{x'^2 + y'^2}\, dt$

$x' = -2\sin 2t \qquad y' = 2\cos 2t.$

$x'^2 + y'^2 = 4(\sin^2 2t + \cos^2 2t) = 4.$

$\therefore \text{Arc length} = \int_0^{\pi/4} 2\, dt = 2t \Big]_0^{\pi/4} = \frac{\pi}{2}.$