SOLUTION

Name: ____________________ ID Number: ____________

UNIVERSITY OF CALGARY

DEPARTMENT OF MATHEMATICS AND STATISTICS

MATHEMATICS 253 — L03 WINTER 04

QUIZ #4a [04-03-22(Monday)]

Max 30 marks (best 3 out of 4 problems).

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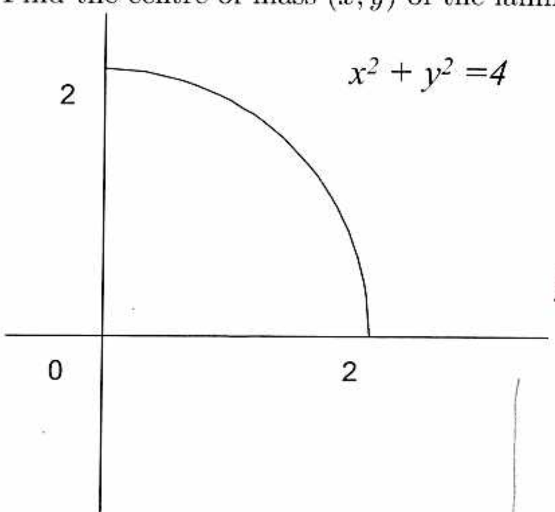
1. Find the centre of mass $(\overline{x}, \overline{y})$ of the lamina shaped as given, assuming its density is 1.



$$y = \sqrt{4-x^2}$$

$$M = \int_0^2 \sqrt{4-x^2}\,dx = \frac{1}{4}\pi r^2 = \pi.$$

$$\bar{x} = \frac{1}{M}\int_0^2 x\sqrt{4-x^2}\,dx = \frac{1}{M}\frac{2(4-x^2)^{3/2}}{3\cdot -2}\Big]_0^2$$

$$= \frac{1}{\pi}\,\frac{-1}{3}(4-x^2)^{3/2}\Big]_0^2$$

$$= \frac{1}{3\pi}\,4^{3/2} = \frac{8}{3\pi}.$$

$$\bar{y} = \frac{1}{M}\,\frac{1}{2}\int_0^2 4-x^2\,dx = \frac{1}{2\pi}\left[4x-\frac{x^3}{3}\right]_0^2$$

$$= \frac{1}{2\pi}\left[8-\frac{8}{3}\right] = \frac{8}{3\pi}.$$

$$\therefore C = \left(\frac{8}{3\pi}, \frac{8}{3\pi}\right)$$

2. Solve $\dfrac{dy}{dx} = y\sqrt{1-x^2}, y(0) = 1$

$$\int_1^y \frac{dy}{y} = \int_0^x \sqrt{1-x^2}\,dx$$

$$\ln|y|\Big]_1^y = \int \cos\theta\cdot\cos\theta\,d\theta$$

Let $x = \sin\theta$

$dx = \cos\theta\,d\theta$

$$\ln|y| - \ln 1 = \frac{1}{2}\int(1+\cos 2\theta)\,d\theta$$

$$= \frac{1}{2}\theta + \frac{1}{2}\sin 2\theta$$

$$= \frac{1}{2}\theta + \sin\theta\cos\theta\Big]$$

$$= \frac{1}{2}\sin^{-1}x + x\sqrt{1-x^2}\Big]_0^x$$

$$\ln|y| = \frac{1}{2}\sin^{-1}x + x\sqrt{1-x^2}$$

or $$y = e^{\frac{1}{2}\sin^{-1}x + x\sqrt{1-x^2}}.$$

3. Solve $x\frac{dy}{dx} - y = 6x^2$

$\frac{dy}{dx} - \frac{y}{x} = 6x.$ linear eq.

$I.F. = e^{-\int \frac{1}{x} dx} = e^{-\ln x} = x^{-1}$

$\therefore x^{-1} y = \int 6x . x^{-1} dx = \int 6 \, dx = 6x + c$

$\therefore y = 6x^2 + cx.$

4. The characteristic roots of a 3rd order differential equation are:

$$m = 1, 1, 3.$$

(i) Find the differential equation.

$(m-1)^2 (m-3) = 0$

$(m^2 - 2m + 1)(m-3) = 0$

$m^3 - 5m^2 + 7m - 3 = 0$

$y''' - 5y'' + 7y' - 3y = 0$

(ii) Find the general solution.

$y = e^x (c_1 + c_2 x) + c_3 e^{3x}.$