Solution

Name: ____________________ ID Number: ____________

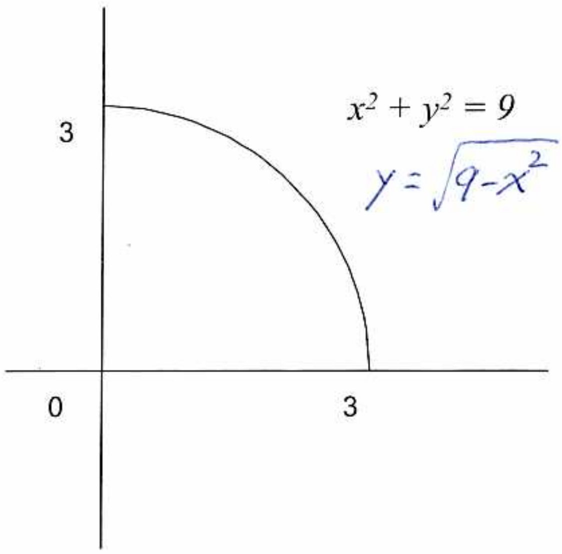
UNIVERSITY OF CALGARY
DEPARTMENT OF MATHEMATICS AND STATISTICS

MATHEMATICS 253 — L03 WINTER 04

QUIZ #4b [04-03-23(Tuesday)]

Max 30 marks (best 3 out of 4 problems).

1. Find the centre of mass $(\overline{x}, \overline{y})$ of the lamina shaped as given, assuming its density is 1.



$$M = \int_0^3 \sqrt{9-x^2}\,dx = \frac{1}{4}\pi r^2 = \frac{9\pi}{4}.$$

$$\bar{x} = \frac{1}{M}\int_0^3 x\sqrt{9-x^2}\,dx$$

$$= \frac{1}{M}\cdot \cancel{\frac{-1}{2}}\,\cancel{\frac{2}{3}}\,(9-x^2)^{3/2}\Big]_0^3$$

$$= \frac{1}{3M}\,9^{3/2} = \frac{27}{3\cdot\frac{9\pi}{4}} = \frac{4}{\pi}.$$

$$\bar{y} = \frac{1}{M}\int_0^3 \frac{1}{2}(9-x^2)\,dx$$

$$= \frac{1}{M}\left[\frac{9x}{4} - \frac{x^3}{6}\right]_0^3$$

$$= \frac{1}{M}\left(\frac{27}{4} - \frac{9}{2}\right) = \frac{1}{\frac{9\pi}{4}}\cdot\frac{45}{4}$$

$$= \frac{5}{\pi}.$$

$$\therefore C = \left(\frac{4}{\pi}, \frac{5}{\pi}\right)$$

2. Solve $\dfrac{dy}{dx} = x\sqrt{1-y^2}, y(0) = 1$

$$\int_1^y \frac{dy}{\sqrt{1-y^2}} = \int_0^x x\,dx$$

$$\sin^{-1} y\Big]_1^y = \frac{x^2}{2}\Big]_0^x$$

$$\sin^{-1} y - \sin^{-1} 1 = \frac{x^2}{2}.$$

$$\sin^{-1} y = \frac{x^2}{2} + \frac{\pi}{2}.$$

$$y = \sin\left(\frac{x^2}{2} + \frac{\pi}{2}\right).$$

3. Solve $x\dfrac{dy}{dx} + y = x^2 \sin x, y(0) = 0.$

$$\frac{dy}{dx} + \frac{y}{x} = x \sin x \qquad \text{linear eq.}$$

$$IF \quad e^{\int \frac{1}{x} dx} = e^{\ln x} = x.$$

$$\therefore \; xy = \int x^2 \sin x \, dx = x^2(-\cos x) + \int 2x \cos x \, dx$$

$$= -x^2 \cos x + 2\left[x \sin x - \int \sin x \, dx\right]$$

$$= -x^2 \cos x + 2x \sin x + 2\cos x + c.$$

To find c:

$$\therefore 0 = \qquad 2 + c \quad \therefore c = -2$$

$$\therefore y = -x\cos x + 2\sin x + \frac{2\cos x}{x} - \frac{2}{x}.$$

4. The characteristic roots of a 3rd order differential equation are:

$$m = -1, -1, 2.$$

(i) Find the differential equation.

$$(m+1)^2(m-2) = 0$$

$$(m^2+2m+1)(m-2) = 0$$

$$m^3 - 3m - 2 = 0$$

$$\therefore y''' - 3y' - 2y = 0$$

(ii) Find the general solution.

$$y = e^{-x}(c_1 + c_2 x) + c_3 e^{2x}.$$