Solution

Name: ______________________ ID Number: ____________

UNIVERSITY OF CALGARY
DEPARTMENT OF MATHEMATICS AND STATISTICS

MATHEMATICS 253 — L03 WINTER 2004

QUIZ #5a [04-04-05(Mon)]

Attempt all problems. Each problem: 10 marks. Total: 30 marks (best 3 out of 4 problems).

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1. Find the orthogonal trajectories of the one-parameter family of curves

$$y^2 = kx$$

$$2yy' = k$$

$$y' = \frac{k}{2y} = \frac{y^2}{2yx} = \frac{y}{2x}$$

$\therefore$ O.T. is :

$$y' = -\frac{2x}{y}$$

$$\int y\,dy = \int -2x\,dx$$

$$\frac{y^2}{2} = -x^2 + c \quad \text{or } \underline{y^2 = -2x^2 + c_1}, \quad c_1 = 2c$$

2. Solve $\quad 2y'' - 5y' + 3y = 3x, \quad y(0) = 0, y'(0) = 1$

$$2m^2 - 5m + 3 = (2m-3)(m-1) = 0 \quad \therefore m = \frac{3}{2}, 1.$$

$$\therefore y_c = c_1 e^{\frac{3}{2}x} + c_2 e^x \qquad \text{U.C. set } \{x, 1\}.$$

$$\therefore y_p = Ax + B \qquad \therefore y_p' = A, \quad y_p'' = 0$$

Hence $\quad -5A + 3(Ax+B) = 3x$

$$\therefore 3A = 3 \qquad \therefore A = 1$$

$$-5A + 3B = 0 \qquad \therefore B = \frac{5}{3}.$$

$$\therefore y = y_c + y_p = c_1 e^{\frac{3}{2}x} + c_2 e^x + x - \frac{5}{3}.$$

$$\therefore y(0) = c_1 + c_2 - \frac{5}{3} = 0 \qquad (1)$$

$$y'(0) = \frac{3}{2}c_1 + c_2 = 0. \qquad (2)$$

$$(1)-(2) \qquad -\frac{1}{2}c_1 \qquad -\frac{5}{3} = 0 \qquad \therefore c_1 = \frac{-10}{3}$$

$$c_2 = -\frac{3}{2}c_1 = +5$$

$$\therefore y = \frac{-10}{3}e^{\frac{3}{2}x} + 5e^x + x - \frac{5}{3}$$

3. Solve $y'' + 4y = 4x + \sin 2x$

$$m^2 + 4 = 0$$

$$m = \pm 2i$$

$$y_c = c_1 \cos 2x + c_2 \sin 2x$$

U.C. set $\{x, 1\} + \{\sin 2x, \cos 2x\}$

$\downarrow$ modify to $\{x \sin 2x, x \cos 2x\}$

$$y = Ax + B + Cx \sin 2x + Dx \cos 2x.$$

$$y' = A + C(\sin 2x + 2x \cos 2x) + D(\cos 2x - 2x \sin 2x)$$

$$y'' = C(2\cos 2x + 2\cos 2x - 4x \sin 2x)$$
$$+ D(-2\sin 2x - 2\sin 2x - 4x \cos 2x).$$

$$\therefore y'' + 4y = 4(Ax + B) + 4C \cos 2x - 4D \sin 2x = 4x + \sin 2x$$

$\therefore 4A = 4 \qquad \therefore A = 1$

$4B = 0 \qquad B = 0$

$4C = 0 \qquad C = 0$

$-4D = 1 \qquad D = -\frac{1}{4}$

$\therefore$ Solution is

$$y = c_1 \cos 2x + c_2 \sin 2x + x - \frac{1}{4} \cdot x \cos 2x.$$

4. Use variation of parameters method to solve

$$y'' + y = \sec x.$$

$m^2+1=0$

$m=\pm i$

$y = c_1(x)\cos x + c_2(x)\sin x.$

$$\therefore c_1' = \frac{\begin{vmatrix} 0 & \sin x \\ \sec x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{-\sin x \sec x}{}$$

$\longrightarrow \cos^2 x + \sin^2 x = 1$

$$\therefore c_1 = -\int \sin x \sec x \, dx = -\int \frac{\sin x}{\cos x} dx$$

$$= \ln \cos x + k_1$$

$$c_2' = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \end{vmatrix}}{1}$$

$$= \cos x \cdot \sec x = 1$$

$$\therefore c_2 = \int 1 \, dx = x + k_2$$

$$\therefore y = (\ln \cos x + k_1)\cos x + (x+k_2)\sin x$$

$$= k_1 \cos x + k_2 \sin x + \cos x(\ln \cos x) + x \sin x.$$