Solution

Name: ______________________ ID Number: ____________

UNIVERSITY OF CALGARY
DEPARTMENT OF MATHEMATICS AND STATISTICS

MATHEMATICS 253 — L03 WINTER 2004

QUIZ #5b [04-04-06(Tue)]

Attempt all problems. Each problem: 10 marks. Total: 30 marks (best 3 out of 4 problems).

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1. Find the orthogonal trajectories of the one-parameter family of curves

$$y = \frac{1}{x+c}$$

$$y' = -\frac{1}{(x+c)^2} = -y^2$$

$\therefore$ O.T. is
$$\frac{dy}{dx} = -\frac{1}{-y^2} = \frac{1}{y^2}.$$

$$\int y^2 dy = \int dx$$

$$\frac{y^3}{3} = x + c.$$

$$y^3 = 3x + c_1, \quad c_1 = 3c.$$

2. Solve $2y'' - y' - y = 2x, \quad y(0) = 0, y'(0) = -2$

$$2m^2 - m - 1 = (2m+1)(m-1) = 0 \qquad \therefore m = -\tfrac{1}{2}, 1$$

$$\therefore y_c = c_1 e^{-\frac{1}{2}x} + c_2 e^{x}. \qquad \text{U.C. set } \{x, 1\}.$$

$$\therefore y_p = Ax + B. \qquad y_p' = A, \quad y_p'' = 0.$$

$$-A - (Ax+B) = 2x.$$

$$\therefore -A = 2, \qquad \therefore A = -2$$

$$-A - B = 0 \qquad B = 2.$$

$$y = y_c + y_p = c_1 e^{-\frac{1}{2}x} + c_2 e^{x} - 2x + 2.$$

$$\therefore y(0) = c_1 + c_2 + 2 = 0 \quad \text{—— (1)}$$

$$y'(0) = -\tfrac{1}{2}c_1 + c_2 - 2 = -2. \quad \text{—— (2)}$$

$$(1) - (2) \qquad \tfrac{3}{2}c_1 + 0 = -2 \qquad \therefore c_1 = -\tfrac{4}{3}, \quad c_2 = -c_1 - 2 = \tfrac{-2}{3}$$

$$\therefore y = -\tfrac{4}{3}e^{-\frac{1}{2}x} - \tfrac{2}{3}e^{x} - 2x + 2$$

3. Solve $y'' - 9y = 9x + e^{3x}$

$$m^2 - 9 = 0 \quad \therefore m = \pm 3.$$

$$y_c = c_1 e^{3x} + c_2 e^{-3x}.$$

U.C. sets $\{x, 1\} + \{e^{3x}\}$

$\downarrow$ modifies $\{xe^{3x}\}$

$$\therefore y_p = Ax + B + Cxe^{3x}$$

$$y_p' = A + C(3xe^{3x} + e^{3x})$$

$$y_p'' = C(9xe^{3x} + 3e^{3x} + 3e^{3x})$$

$$= C(9xe^{3x} + 6e^{3x})$$

$$\therefore y_p'' - 9y_p = -9(Ax+B) + 6Ce^{3x} = 9x + e^{3x}.$$

$$\therefore -9A = 9$$
$$-9B = 0$$
$$6C = 1$$

$$\therefore A = -1, \quad B = 0, \quad C = \frac{1}{6}.$$

$$\therefore y = c_1 e^{3x} + c_2 e^{-3x} - x + \frac{1}{6} x e^{3x}.$$

4. Use variation of parameters method to solve

$$y'' + 4y = \sec 2x.$$

$$m^2 + 4 = 0 \quad \therefore m = \pm 2i$$

$$y = c_1(x)\cos 2x + c_2(x)\sin 2x$$

$$c_1' = \frac{\begin{vmatrix} 0 & \sin 2x \\ \sec 2x & 2\cos 2x \end{vmatrix}}{\begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}} = \frac{-\sin 2x \sec 2x}{}$$

$$\longrightarrow 2\cos^2 2x + 2\sin^2 2x = 2$$

$$c_1 = -\frac{1}{2}\int \sin 2x \cdot \frac{1}{\cos 2x}\, dx = \frac{1}{4}\ln \cos 2x + k_1$$

$$c_2' = \frac{\begin{vmatrix} \cos 2x & 0 \\ -2\sin 2x & \sec 2x \end{vmatrix}}{2} = \frac{1}{2}\cos 2x \cdot \sec 2x$$

$$c_2 = \int \frac{1}{2}\, dx = \frac{x}{2} + k_2$$

$$\therefore y = \left(\frac{1}{4}\ln \cos 2x + k_1\right)\cos 2x + \left(\frac{x}{2} + k_2\right)\sin 2x$$

$$= k_1 \cos 2x + k_2 \sin 2x + \frac{1}{4}(\ln \cos 2x)\cos 2x + \frac{x}{2}\sin 2x$$