UNIVERSITY OF CALGARY

DEPARTMENT OF MATHEMATICS AND STATISTICS

MATHEMATICS 253 — L03 WINTER 2004

MIDTERM EXAM [04-03-03(Wed)]

Time: 50 minutes. PLEASE write your Name on the very last page.

Student ID: ________________

[Marks]

[4] 1. (a) Find the inverse function of $f(x) = \dfrac{1-x}{1-2x}$ and find its domain and range:

Inverse fn:

$$x = \frac{1-y}{1-2y}.$$

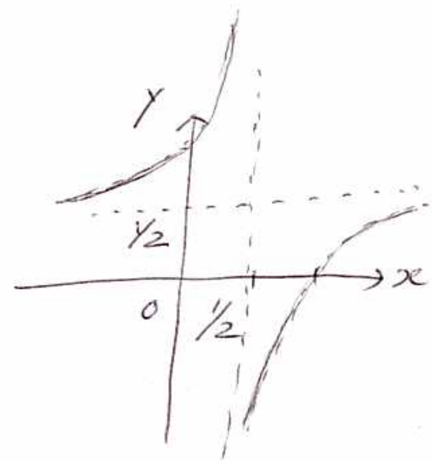
$$x(1-2y) = 1-y.$$

$$-2xy + y = 1-x.$$

$$y = \frac{1-x}{1-2x}.$$

Domain: $x \neq \frac{1}{2}$.

Range: $y \neq \frac{1}{2}$.



[2] (b) Evaluate $\dfrac{d}{dx}\displaystyle\int_{2x}^{x^2} \sin t^2 \, dt.$

$$= 2x \sin x^4 - 2 \sin 4x^2.$$

[4] 2. Evaluate $\int_1^e x^2 \ln x \, dx.$ (by parts)

$$= \frac{x^3}{3} \ln x \Big]_1^e - \int_1^e \frac{x^3}{3} \cdot \frac{1}{x} \, dx$$

$$= \frac{x^3}{3} \ln x - \frac{x^3}{9} \Big]_1^e$$

$$= \frac{e^3}{3} - \frac{e^3}{9} + \frac{1}{9}$$

$$= \frac{2e^3+1}{9}$$

[5] 3. Evaluate $\displaystyle\int \frac{x}{x^2-2x+5}\,dx.$

$$x^2-2x+5 = (x-1)^2-1+5$$
$$= (x-1)^2+4.$$

$$= \int \frac{x\,dx}{(x-1)^2+4}$$

Let $u = x-1$
$du = dx$.

$$= \int \frac{u+1}{u^2+4}\,du$$

$$= \int \frac{u}{u^2+4} + \frac{1}{u^2+4}\,du$$

$$= \frac{1}{2}\ln|u^2+4| + \frac{1}{2}\tan^{-1}\frac{u}{2} + c$$

$$= \frac{1}{2}\ln\left(x^2-2x+5\right) + \frac{1}{2}\tan^{-1}\frac{x-1}{2} + C.$$

[5] 4. Evaluate $\int \sqrt{1-4x^2}\,dx$.

Let $2x = \sin\theta$.

$x = \frac{1}{2}\sin\theta$

$dx = \frac{1}{2}\cos\theta\, d\theta$.

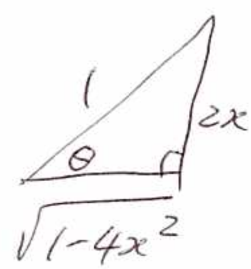
$$= \int \cos\theta \cdot \frac{1}{2}\cos\theta\, d\theta$$

$$= \frac{1}{2}\int \cos^2\theta\, d\theta$$

$$= \frac{1}{2}\cdot\frac{1}{2}\int 1+\cos 2\theta\, d\theta.$$

$$= \frac{1}{4}\left[\theta + \frac{1}{2}\sin 2\theta\right] + C$$

$$= \frac{1}{4}\left[\theta + \sin\theta\cos\theta\right] + C$$



$$= \frac{1}{4}\left[\sin^{-1}(2x) + 2x\sqrt{1-4x^2}\right] + C$$

[5] 5. Evaluate the improper integral $\int_0^\infty xe^{-2x}\,dx$.

$$= \lim_{N\to\infty} \int_0^N xe^{-2x}\,dx \qquad \text{(by parts)}$$

$$= \quad '' \quad \left.\frac{x}{-2}e^{-2x}\right]_0^N + \frac{1}{2}\int_0^N 1.e^{-2x}\,dx$$

$$= \quad '' \quad \left.\frac{x}{-2}e^{-2x} - \frac{1}{4}e^{-2x}\right]_0^N$$

$$= \quad '' \quad \frac{Ne^{-2N}}{-2} - \frac{1}{4}e^{-2N} + \frac{1}{4}e^{0}.$$

By l'Hopital's rule.

$$\lim_{N\to\infty} \frac{N}{e^{2N}} = \lim_{N\to\infty} \frac{1}{2e^{2N}} = 0$$

$$= \frac{1}{4}.$$

[5] 6. Find the arc length of the curve $y = \sqrt{1-x^2}$ from $x = 0$ to $x = 1$.

$$\text{Arc length} = \int_0^1 \sqrt{1+y'^2}\, dx$$

$$y' = \frac{-2x}{2\sqrt{1-x^2}} = \frac{-x}{\sqrt{1-x^2}}$$

$$y'^2 = \frac{x^2}{1-x^2}$$

$$1+y'^2 = 1 + \frac{x^2}{1-x^2}$$

$$= \frac{1-x^2+x^2}{1-x^2}$$

$$= \frac{1}{1-x^2}$$

$$= \int_0^1 \frac{1}{\sqrt{1-x^2}}\, dx.$$

$$= \sin^{-1} x \Big]_0^1$$

$$= \sin^{-1} 1 - \sin^{-1} 0$$

$$= \frac{\pi}{2}.$$