Winter '04 MATH 253 (L03)

Final Review Solution

1 (i) $\displaystyle\int \frac{x}{(x-2)(x+1)}\,dx$

$$\frac{x}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$$

$$x = A(x+2) + B(x-2)$$

$$\therefore 1 = A + B.$$

$$0 = 2A - 2B.$$

Hence $A = B$

$$\therefore 1 = 2A \quad \therefore A = \tfrac{1}{2},\ B = \tfrac{1}{2}.$$

$$= \int \frac{\frac{1}{2}}{x-2} + \frac{\frac{1}{2}}{x+1}\,dx$$

$$= \frac{1}{2}\ln|x-2| + \frac{1}{2}\ln|x+1| + C$$

or $\frac{1}{2}\ln|x^2 - x - 2| + C.$

(ii) $\displaystyle\int \frac{\sqrt{4x^2-1}}{x}\,dx$.

Let $2x = \sec\theta$ or $x = \frac{1}{2}\sec\theta.$

$dx = \frac{1}{2}\sec\theta\, d\theta$

$$\int \frac{\tan\theta}{\frac{1}{2}\sec\theta}\cdot\frac{1}{2}\sec\theta\, d\theta = \int \tan\theta\, d\theta$$

$$= \ln \sec\theta + C$$

$$= \ln 2x + C$$

$$= \ln 2 + \ln x + C$$

$$= \ln x + C_1, \quad C_1 = \ln 2 + C$$

(iii) $\displaystyle\int x^3 e^{-3x}\,dx$

$$= x^3\frac{e^{-3x}}{-3} + \frac{1}{\cancel{3}}\int \cancel{3}x^2 e^{-3x}\,dx$$

$$= \text{''} + x^2\frac{e^{-3x}}{-3} + \frac{1}{3}\int 2x\, e^{-3x}\,dx$$

$$= \text{''} + \text{''} + \frac{2}{3}\left[x\frac{e^{-3x}}{-3} + \frac{1}{3}\int e^{-3x}\,dx\right]$$

$$= \text{''} + \text{''} + \frac{2}{3}\left[\text{''} - \frac{1}{9}e^{-3x}\right] + C$$

$$= e^{-3x}\left[-\frac{x^3}{3} - \frac{x^2}{-3} - \frac{2x}{9} - \frac{2}{27}\right] + C.$$

(iv) $\int \sqrt{1+4x^2}\,dx$ Let $2x = \tan\theta$

$x = \frac{1}{2}\tan\theta$

$= \int \sec\theta \cdot \frac{1}{2}\sec^2\theta\, d\theta$ $dx = \frac{1}{2}\sec^2\theta\, d\theta$

$= \frac{1}{2}\int \sec^3\theta\, d\theta.$ Let $I = \int \sec^3\theta\, d\theta$ (Or use table,

$= \int \sec^2\theta \sec\theta\, d\theta$

$= \frac{1}{4}\left(\tan\theta\sec\theta + \ln|\sec\theta+\tan\theta|\right)+C$

$= \tan\theta\cdot\sec\theta - \int \tan\theta\sec\theta\tan\theta\, d\theta$

$= \tan\theta\sec\theta - \int(\sec^2\theta-1)\sec\theta\, d\theta$

$=$ " $- I + \int \sec\theta\, d\theta.$

$+ \ln|\sec\theta+\tan\theta|$

$\therefore 2I =$ "

$\therefore I = \frac{1}{2}[$ " $+$ " $]$

(v) $\int \sqrt{1+\frac{x^2}{9}}\,dx$

Try $x = 3\tan\theta$

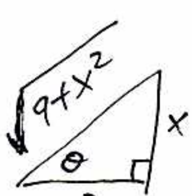
(vi) $\int x\sin^{-1}2x\,dx = \frac{x^2}{2}\sin^{-1}2x - \frac{1}{2}\int x^2 \cdot \frac{2}{\sqrt{1-4x^2}}\,dx$

$=$ " $-\int \frac{x^2}{\sqrt{1-4x^2}}\,dx.$

Let $2x = \sin\theta$

$x = \frac{1}{2}\sin\theta$, $dx = \frac{1}{2}\cos\theta\, d\theta.$

$=$ " $-\int \frac{\frac{1}{4}\sin^2\theta}{\cancel{\cos\theta}}\frac{1}{2}\cancel{\cos\theta}\, d\theta$



$=$ " $-\frac{1}{8}\int \frac{1}{2}(1-\cos 2\theta)\, d\theta$

$=$ " $-\frac{1}{16}\left(\theta - \frac{1}{2}\sin 2\theta\right) + C.$

$\frac{1}{2}\sin 2\theta = \frac{2}{2}\sin\theta\cos\theta$

$= \frac{x^2}{2}\sin^{-1}2x - \frac{1}{16}\left(\tan^{-1}\frac{x}{3} - \frac{3x}{9+x^2}\right) + C$

(vii) $\int x \tan^{-1} 2x \, dx = \frac{x^2}{2} \tan^{-1} 2x - \frac{1}{2} \int x^2 \cdot \frac{2}{1+4x^2} \, dx$

$= \quad '' \quad - \int \frac{x^2}{1+4x^2} \, dx.$

$= \quad '' \quad - \int \frac{1}{4} - \frac{1}{4} \frac{1}{1+4x^2} \, dx.$

$= \quad '' \quad - \frac{x}{4} + \frac{1}{4} \frac{1}{2} \tan^{-1} 2x + C$

$= \quad '' \quad - \frac{x}{4} + \frac{1}{8} \tan^{-1} 2x + C$

(viii) $\int_1^3 \frac{dx}{(3-x)^{1/2}}$ (improper integral)

$= \lim_{\delta \to 0^+} \int_1^{3-\delta} \frac{dx}{(3-x)^{1/2}}$

$= \quad '' \quad -2(3-x)^{\frac{1}{2}} \Big]_1^{3-\delta}$

$= \quad '' \quad -2(\delta)^{\frac{1}{2}} + 2 \cdot 3^{\frac{1}{2}} = 2\sqrt{3}.$

(ix) $\int \sqrt{x} \sqrt{1 + \frac{1}{x}} \, dx$

$= \int \sqrt{x} \sqrt{\frac{x+1}{x}} \, dx$

$= \int \sqrt{x+1} \, dx = \frac{2}{3}(x+1)^{3/2} + C$

2 (i) Arc length $= \int_1^2 \sqrt{1+y'^2} \, dx = \int_1^2 \frac{\sqrt{x^2+1}}{x} \, dx.$

$y = \ln x$

$y' = \frac{1}{x}.$

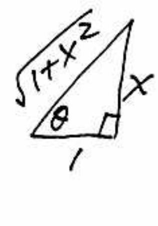
$y'^2 = \frac{1}{x^2}$

$1 + y'^2 = 1 + \frac{1}{x^2}$

$= \frac{x^2+1}{x^2}$

Let $x = \tan\theta$

$dx = \sec^2\theta \, d\theta$



$= \int \frac{\sec\theta}{\tan\theta} \sec^2\theta \, d\theta$

$= \int \frac{\sec\theta}{\tan\theta} (\ ^2\theta + 1) \, d\theta = \int \sec\theta \tan\theta + \frac{\sec\theta}{\tan\theta}$

$= \sec\theta + \int \frac{1}{\cos\theta} \cdot \frac{\cos\theta}{\sin\theta} \, d\theta$

$= \quad '' \quad - \ln|\operatorname{cosec}\theta - \cot\theta|$

$= \sqrt{1+x^2} - \ln\left|\sqrt{1+x^2} - 1\right| \Big]^2$ etc

2 (ii) Arc length $= \int \sqrt{1+y'^2}\, dx$ or $\int_0^2 \sqrt{1+x'^2}\, dy$

$y^2 = 4x$

$\frac{d}{dy}$: $2y = 4\frac{dx}{dy}$

$\frac{dx}{dy} = \frac{y}{2}$

$= \int_0^2 \sqrt{1+\frac{y^2}{4}}\, dy$

Similar to 1. (iv).

3. Pg 559 prob. 12

Surface area $= 2\pi \int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$.

or $2\pi \int_c^d y\sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$

In this case, $x = 1+2y^2$

$\therefore \frac{dx}{dy} = 4y$

$S = 2\pi \int_1^2 y\sqrt{1+16y^2}\, dy$

$= \frac{2\pi}{32} \int_1^2 32y\,(1+16y^2)^{\frac{1}{2}}\, dy$

$= \frac{\pi}{16} \cdot \frac{2}{3}(1+16y^2)^{\frac{3}{2}}\Big]_1^2$

$= \frac{\pi}{24}\left[(1+16\times 4)^{3/2} - (17)^{3/2}\right]$

$= \frac{\pi}{24}\left((65)^{3/2} - (17)^{3/2}\right).$

4. Pg 607 prob 11.

$$\frac{dy}{y^2+1} = dx$$

$\therefore \tan^{-1} y = x + c.$

Using $y(1)=0$, we get

$\tan^{-1} 0 = 1 + c \quad \therefore c = -1$

$\therefore \tan^{-1} y = x-1 \quad$ or $y = \tan(x-1).$

prob. 18.

$xy' = y^2 - y \qquad$ Separable

$\therefore \frac{dy}{y^2-y} = \frac{dx}{x} \qquad \frac{1}{y^2-y} = \frac{1}{y(y-1)} = \frac{A}{y} + \frac{B}{y-1}.$

$1 = A(y-1) + By$

$\int \frac{-1}{y} + \frac{1}{y-1}\, dy = \int \frac{dx}{x}$

Set:

$y=0 \quad 1 = -A \quad \therefore A = -1$

$y=1 \quad 1 = B$

$-\ln y + \ln y-1 = \ln x + \ln c$

$\ln \frac{y-1}{yc} = \ln x$

$\frac{y-1}{yc} = x.$

$y-1 = cyx.$

At $x=1$,

$-1-1 = -c \qquad \therefore c = 2$

$\therefore y-1 = 2yx.$

5 Pg 636 prob. 7.

$$xy' - 2y = x^2$$

$$y' - \frac{2}{x}y = x \qquad \text{Linear}$$

IF $e^{-\int \frac{2}{x}dx} = e^{-2\ln x} = x^{-2}$.

Hence the general solution is

$$x^{-2}y = \int x^{-2}x^2\,dx + c$$

$$x^{-2}y = x + c$$

$$\therefore y = x^3 + cx^2.$$

Prob 20

$$\frac{dy}{dx} - \frac{y}{x(x+1)} = 1 \quad , \quad y(1) = 0$$

IF $e^{-\int \frac{1}{x(x+1)}dx}$ $= e^{-\int \frac{1}{x} - \frac{1}{x+1}\,dx}$

$$= e^{-\ln x + \ln x+1}$$

$$= e^{\ln \frac{x+1}{x}} = \frac{x+1}{x}.$$

$$\therefore \frac{x+1}{x}y = \int \frac{x+1}{x}dx = \int 1 + \frac{1}{x}\,dx$$

$$= x + \ln x + c$$

$$\therefore y = \frac{x}{x+1}\left(\quad '' \quad \right).$$

At $x = 1$,

$$0 = \frac{1}{2}(1 + c) \qquad \therefore c = -1$$

$$\therefore y = \frac{x}{x+1}(x + \ln x - 1).$$

6. (i) $y' = xy - 3x$

Method A: $\quad y' = x(y-3)$ Separable

$$\frac{dy}{y-3} = x\,dx.$$

$$\ln \frac{y-3}{c} = \frac{x^2}{2}$$

$$\therefore \frac{y-3}{c} = e^{\frac{x^2}{2}}$$

$$y - 3 = ce^{\frac{x^2}{2}}, \text{ where } c = \text{arb. const.}$$

Method B

$$y' - xy = -3x \quad \text{Linear d. eq.}$$

IF $\quad e^{-\int x\,dx} = e^{-\frac{x^2}{2}}$

$\therefore$ The solution is

$$e^{-\frac{x^2}{2}} y = -3\int x e^{-\frac{x^2}{2}} dx + c$$

$$= -3 \cdot -e^{-\frac{x^2}{2}} + c$$

$$\therefore y = +3 + ce^{\frac{x^2}{2}}, \text{ as before.}$$

6 (ii) $2y' - x^2 y = 3x^2$

Method A $\qquad 2y' = x^2(3+y) \qquad$ Separable.

$$\frac{2\,dy}{3+y} = x^2\,dx$$

$$2 \ln \frac{3+y}{c} = \frac{x^3}{3}.$$

$$\ln \frac{3+y}{c} = \frac{x^2}{6}$$

$$\therefore 3+y = ce^{\frac{x^2}{6}}.$$

Method B

$$y' - \frac{x^2}{2}y = \frac{3}{2}x^2 \qquad \text{Linear.}$$

$$\text{I.F.} = e^{-\frac{1}{2}\int x^2\,dx} = e^{-\frac{x^3}{6}}$$

$\therefore$ The general solution is

$$e^{-\frac{x^3}{6}}y = \frac{3}{2}\int x^2 e^{-\frac{x^3}{6}}\,dx$$

$$= 3\int \frac{x^2}{2}\, e^{-\frac{x^3}{6}}\,dx \qquad \int f' e^f\,dx = e^f.$$

$$= 3 \cdot -e^{-\frac{x^3}{6}} + c$$

$$\therefore y = -3 + ce^{\frac{x^3}{6}}, \text{ as above.}$$

7 Pg 1147 prob 18

$$y'' + 3y = 0, \quad y(0) = 1, \quad y'(0) = 3$$

$$m^2 + 3 = 0 \quad \therefore m = \pm\sqrt{3}\,i$$

$$y = c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x$$

$$y' = -\sqrt{3}\,c_1 \sin\sqrt{3}x + \sqrt{3}\,c_2 \cos\sqrt{3}x$$

At $x = 0$,

$$1 = c_1 \cos 0 + c_2 \sin 0 \quad = c_1$$

$$3 = -\sqrt{3}\,c_1 \sin 0 + \sqrt{3}\,c_2 \cos 0 = \sqrt{3}\,c_2.$$

$$\therefore c_1 = 1, \quad c_2 = \frac{3}{\sqrt{3}} = \sqrt{3}.$$

$$\therefore y = \cos\sqrt{3}x + \sqrt{3}\sin\sqrt{3}x.$$

8 Pg 1154, prob 8

$$y'' - 4y = e^x \cos x, \quad y(0) = 1, \quad y'(0) = 2.$$

$$m^2 - 4 = 0 \quad \therefore m = \pm 2.$$

$$y_c = c_1 e^{2x} + c_2 e^{-2x}.$$

U.C. set $\{e^x \cos x, \; e^x \sin x\}$

$$y_p = Ae^x \cos x + Be^x \sin x. \qquad \text{(Chain rule)}$$

$$y_p' = Ae^x(\cos x - \sin x) + Be^x(\sin x + \cos x)$$

$$y_p'' = Ae^x(\cos x - \sin x - \sin x - \cos x) + Be^x(\sin x + \cos x + \cos x - \sin x)$$

$$= -2Ae^x \sin x \qquad + 2Be^x \cos x$$

$$\therefore y_p'' - 4y_p = -2Ae^x \sin x + 2Be^x \cos x - 4(Ae^x \cos x + Be^x \sin x)$$

$$= (-2A - 4B)e^x \sin x + (2B - 4A)e^x \cos x = e^x \cos x$$

$\therefore \quad -2A - 4B = 0 \qquad \therefore A = -2B$

$2B - 4A = 1 \qquad 2B + 8B = 1$

$\therefore B = \frac{1}{10}$

$\& \ A = \frac{-1}{5}$

$$\therefore y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{5} e^x \cos x + \frac{1}{10} e^x \sin x.$$

$$y' = 2c_1 e^{2x} - 2c_2 e^{-2x} - \frac{1}{5} e^x (\cos x - \sin x) + \frac{1}{10} e^x (\sin x + \cos x)$$

At $x=0$, we get

$$1 = c_1 + c_2 - \frac{1}{5} \quad \text{———} \quad (1)$$

$$2 = 2c_1 - 2c_2 - \frac{1}{5} + \frac{1}{10} \quad \text{———} \quad 1 = c_1 - c_2 - \frac{1}{20} \quad (2)$$

(1) + (2)

$$2 = 2c_1 - \frac{1}{5} - \frac{1}{20}$$

$$\therefore 2c_1 = 2 + \frac{1}{5} + \frac{1}{20} = \frac{40+4+1}{20} = \frac{45}{20}$$

$$c_1 = \frac{45}{40} = \frac{9}{8}$$

$$\& \ c_2 = 1 - c_1 + \frac{1}{5} = 1 - \frac{9}{8} + \frac{1}{5} = \frac{3}{40}.$$

$$\therefore y = \frac{9}{8} e^{2x} + \frac{3}{40} e^{-2x} - \frac{1}{5} e^x \cos x + \frac{1}{10} e^x \sin x.$$