The University of Calgary
Department of Mathematics and Statistics
MATH 253
Handout # 1-solution

A

1. For $x > 0$ find $\int \sqrt{x}\left(\frac{5}{\sqrt{x}} - \frac{4}{x^{\frac{3}{2}}}\right) dx.$

2. Evaluate $\int\limits_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos^3 x}{\sin^3 x} dx.$

3. Find the inverse function f^{-1} and its range and domain if $f(x) = e^{\sqrt{4-x}}$.

Solutions

For 1)

For $x > 0$ $\int \sqrt{x}\left(\frac{5}{\sqrt{x}} - \frac{4}{x^{\frac{3}{2}}}\right) dx = \int \frac{5\sqrt{x}}{\sqrt{x}} dx - \int \frac{4x^{\frac{1}{2}}}{x^{\frac{3}{2}}} dx =$

$= 5\int dx - 4\int x^{-1} dx = 5x - 4\ln x + c, x > 0$

For 2)

use substitution $u = \sin x, du = \cos x\ dx, \cos^2 x = 1 - \sin^2 x$

if $x = \frac{\pi}{4}$, then $u = \frac{1}{\sqrt{2}}$,and if $x = \frac{\pi}{2}, u = 1$, so

$$\int\limits_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos^3 x}{\sin^3 x} dx = \int\limits_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos^2 x}{\sin^3 x} \cdot \cos x dx = \int\limits_{\frac{1}{\sqrt{2}}}^{1} \frac{(1-u^2)}{u^3} du = \int\limits_{\frac{1}{\sqrt{2}}}^{1} \frac{1}{u^3} du - \int\limits_{\frac{1}{\sqrt{2}}}^{1} \frac{1}{u} du$$

$= \left[\frac{u^{-2}}{-2}\right]_{\frac{1}{\sqrt{2}}}^{1} - [\ln|u|]_{\frac{1}{\sqrt{2}}}^{1} = \frac{-1}{2}[1-2] - \left[\ln 1 - \ln 2^{-\frac{1}{2}}\right] = \frac{1}{2} - \left[0 + \frac{1}{2}\ln 2\right] = \frac{1}{2} - \frac{1}{2}\ln 2.$

For 3)

$D_f = (-\infty, 4]$ since we must have $(4 - x) \geq 0$ and $R_f \subset (0, +\infty)$
since the exponential function has always positive values.
now to find the inverse solve for x: $y = e^{\sqrt{4-x}}$,
if y is positive we can apply ln to both sides
$\ln y = \sqrt{4-x}$ and again the right-hand side is positive or 0
so IF $\ln y \geq 0$ we can square both sides $(\ln y)^2 = 4 - x$ and $x = 4 - \ln^2 y$,
finally by interchanging x and y we get the new formula
$y = f^{-1}(x) = 4 - \ln^2 x, D_{f^{-1}} = R_f =?$,and $R_{f^{-1}} = D_f = (-\infty, 4]$
to get the range of f we have to solve when $\ln y \geq 0$ and that is for $y \geq 1$
from the graph of ln OR by applying exp. function to the inequality and using $e^0 = 1$
thus $R_f = D_{f^{-1}} = [1, +\infty)$.

B

4. For $x \neq 0$ find $\int \frac{5x - \sqrt[3]{x} + 3}{\sqrt[3]{x}} dx$.

5. Evaluate $\int\limits_0^2 \frac{x^2}{3-x} dx$.

6. Find the inverse function f^{-1} and its range and domain if $f(x) = \ln\left(\frac{1}{1-x}\right)$.

For 1)

For $x \neq 0$ $\int \frac{5x - \sqrt[3]{x} + 3}{\sqrt[3]{x}} dx = \int \left(\frac{5x}{\sqrt[3]{x}} - 1 + \frac{3}{x^{\frac{1}{3}}} \right) dx = 5 \int x^{\frac{2}{3}} dx - \int dx + 3 \int x^{-\frac{1}{3}} dx =$
$= 5 \cdot \frac{3}{5} x^{\frac{5}{3}} - x + 3 \cdot \frac{3}{2} x^{\frac{2}{3}} + c = 3x^{\frac{5}{3}} - x + \frac{9}{2} x^{\frac{2}{3}} + c.$

For 2)

use substitution $u = 3 - x, du = -\, dx, x = 3 - u$ and $x^2 = (3-u)^2$
if $x = 0$, then $u = 3$, and if $x = 2, u = 1$, so

$$\int\limits_0^2 \frac{x^2}{3-x} dx = -\int\limits_3^1 \frac{(3-u)^2}{u} du = \int\limits_1^3 \frac{9 - 6u + u^2}{u} du = \int\limits_1^3 \left(\frac{9}{u} - 6 + u \right) du =$$

$= 9 [\ln u]_1^3 - 6 [u]_1^3 + \frac{1}{2} [u^2]_1^3 = 9 [\ln 3 - \ln 1] - 6 [3 - 1] + \frac{1}{2} [9 - 1] = 9 \ln 3 - 12 + 4 = 9 \ln 3 - 8.$

For 3)

$D_f =]-\infty, 1[$ since we must have $\frac{1}{1-x} > 0$ so $1 - x > 0$ thus $1 > x$
and $R_f =]-\infty, +\infty[$ since logarithmic function has positive and negative values.
now to find the inverse solve for x: $y = \ln \frac{1}{1-x} = -\ln(1-x)$,
$-y = \ln(1-x)$ apply exp. function to both sides -possible for any y

$e^{-y} = 1 - x$ and $x = 1 - e^{-y}$ finally by interchanging x and y
(OR $e^y = \frac{1}{1-x}$ $\frac{1}{e^y} = e^{-y} = 1 - x$)
$y = f^{-1}(x) = 1 - e^{-x}$, $D_{f^{-1}} = R_f =]-\infty, +\infty[$,and $R_{f^{-1}} = D_f =](-\infty, 1[$.

C

7. For $x > 0$ find $\int \left(2\sqrt{x} - \frac{1}{x}\right)^2 dx.$

8. Evaluate $\int\limits_{\frac{1}{2}}^{1} \frac{3^{\frac{1}{x}}}{x^2} dx.$

9. Find the inverse function f^{-1} and its range and domain if $f(x) = \dfrac{1-2x}{x+3}$.

   Solutions

For 1)

$$\int \left(2\sqrt{x} - \tfrac{1}{x}\right)^2 dx = \int (2\sqrt{x})^2 - 2\left(2\sqrt{x}\cdot\frac{1}{x}\right) + \left(\frac{1}{x}\right)^2 dx =$$

$$= 4\int x\ dx - 4\int x^{-\frac{1}{2}}dx + \int x^{-2}dx = 4\cdot\tfrac{x^2}{2} - 4\cdot 2x^{\frac{1}{2}} - x^{-1} + c =$$

$$= 2x^2 - 8\sqrt{x} - \frac{1}{x} + c, x > 0$$

For 2)

use substitution $u = \frac{1}{x}, du = -x^{-2}dx$,so $\quad -du = \dfrac{dx}{x^2}$ and if $x = \frac{1}{2}$, then $u = 2$, and if $x = 1, u = 1$ so

$$\int\limits_{\frac{1}{2}}^{1} \frac{3^{\frac{1}{x}}}{x^2} dx = -\int\limits_{2}^{1} 3^u du = \int\limits_{1}^{2} 3^u du = \left[\frac{3^u}{\ln 3}\right]_1^2 = \tfrac{1}{\ln 3}\left[9-3\right] = \frac{6}{\ln 3}.$$

For 3)

$D_f = \{x \neq -3\}$ but $R_f = ?$, now to find the inverse solve for x: $y = \dfrac{1-2x}{x+3}$

for $x \neq -3 \quad y(x+3) = 1 - 2x, \qquad xy + 3y = 1 - 2x \qquad xy + 2x = 1 - 3y,$

so $\quad x(y+2) = 1 - 3y$
to be able to solve for x we have to assume that $(y+2) \neq 0,$

so $y \neq -2$.Therefore $R_f = \{y \neq -2\}$ and we can finish solving : $x = \dfrac{1-3y}{y+2}$
(interchange $x \to y$)

and $\quad f^{-1}(x) = \dfrac{1-3x}{x+2}, D_{f^{-1}} = R_f = \{x \neq -2\}$,and $R_{f^{-1}} = D_f = \{y \neq -3\}$

D

10. Find $\int x\cos(3x^2+1)dx$.

11. Evaluate $\int\limits_{e}^{e^3} \frac{1}{x\ln x}dx$.

12. Find the inverse function f^{-1} and its range and domain if $f(x) = -\sqrt{1+x}$.

Solutions

For 1)

by subst. $u = 3x^2+1 \qquad du = 6xdx$

$\int x\cos(3x^2+1)dx = \frac{1}{6}\int \cos u du = \frac{1}{6}\sin u + c = \frac{1}{6}\sin(3x^2+1)+c$

For 2)

use substitution $u = \ln x, du = \frac{1}{x}dx$, and if $x = e$, then $u = 1$,

and if $x = e^3, u = 3$ since $\qquad \ln e^r = r \qquad$ so

$$\int\limits_{e}^{e^3} \frac{1}{x\ln x}dx = \int\limits_{1}^{3} \frac{1}{u}du = [\ln u]_1^3 = \ln 3 - \ln 1 = \ln 3.$$

For 3)

$D_f = [-1,+\infty)$since we must have $(1+x) \geq 0$ and $R_f = (-\infty, 0]$,

now to find the inverse solve for x: $y = -\sqrt{1+x}$,

$-y = \sqrt{1+x}$, so $-y$ must be positive or 0

$i.e\ .y \leq 0$, then we can square both sides and $y^2 = 1+x$, and $x = y^2-1$,so

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$f^{-1}(x) = x^2-1, D_{f^{-1}} = R_f = (-\infty, 0]$ only,and $R_{f^{-1}} = D_f = [-1,+\infty)$.