MATH 253
Handout # 3

1. **SOLUTION**

For 1)

$\int_1^e x(\ln x)^2 dx =$ (by parts: $du = xdx$..integrate..$u = \frac{x^2}{2}$,

$v = \ln^2 x$..differentiate..$dv = 2\ln x \cdot \frac{1}{x}dx$)

$= \left[\frac{x^2}{2}(\ln x)^2\right]_1^e - \int_1^e \frac{x^2}{2}2(\ln x)\frac{1}{x}dx = \frac{e^2}{2}\cdot 1 - 0 - \int_1^e x(\ln x)dx =$

(by parts again: $du = xdx$..$u = \frac{x^2}{2}, v = \ln x$...$dv = \frac{1}{x}dx$)

$=\frac{e^2}{2} - \left[\frac{x^2}{2}\ln x\right]_1^e + \int_1^e \frac{x^2}{2}\cdot\frac{1}{x}dx = \frac{e^2}{2} - \frac{e^2}{2} + 0 + \frac{1}{2}\left[\frac{x^2}{2}\right]_1^e = \frac{1}{4}\left[e^2 - 1\right].$

For 2)

Use inverse subst. $\quad u = 2 + \sqrt[3]{x} \quad (u-2)^3 = x \quad 3(u-2)^2 du = dx$

$$\int \frac{1}{2+\sqrt[3]{x}}dx = \int \frac{3(u-2)^2}{u}du = 3\int \frac{u^2 - 4u + 4}{u}du = 3\int udu - 12\int du + 12\int \tfrac{1}{u}du =$$

$= \frac{3}{2}u^2 - 12u + 12\ln|u| + c \quad$ (for any $u \neq 0$)

$= \frac{3}{2}\left(2+\sqrt[3]{x}\right)^2 - 12\left(2+\sqrt[3]{x}\right) + 12\ln\left|2+\sqrt[3]{x}\right| + c$

and for the domain

$u \neq 0$ implies that $x \neq -8$ since $(u-2)^3 = x$.

For 3)

Complete the square: $-(x^2 - 6x - 7) = -\left[(x-3)^2 - 16\right],$

so it is a parabola open down

with two roots $x = -1, 7$ thus domain is $(-1, 7)$

by substitution $u = x - 3, dx = du, x = u + 3$,we get

$$\int \frac{x}{\sqrt{7+6x-x^2}}\,dx = \int \frac{3+u}{\sqrt{16-u^2}}\,du = 3\int \frac{1}{\sqrt{16-u^2}}du + \int \frac{u}{\sqrt{16-u^2}}du =$$

$= 3\arcsin\frac{u}{4} - \frac{1}{2}\int \frac{dv}{\sqrt{v}}$ (using $v = 16 - u^2, dv = -2u\ du \qquad u \in (-4,4)) =$

$= 3\arcsin\frac{u}{4} - \sqrt{v} + c = 3\arcsin\frac{x-3}{4} - \sqrt{16-u^2} + c =$

$= 3\arcsin\frac{x-3}{4} - \sqrt{7+6x-x^2} + c \qquad$ for $x \in (-1,7)$

For 4)

by parts:

$du = 1dx$.by integrating $u = x, v = \arctan 2x$..by differentiating $dv = \dfrac{2dx}{1+(2x)^2}$

$$F(x) = \int \arctan(2x) \cdot 1dx = x \cdot \arctan(2x) - \int x \cdot \frac{2}{1+4x^2}dx =$$

$$= x\arctan 2x - \int \frac{2xdx}{1+4x^2} = x\arctan 2x - \tfrac{1}{4}\int \frac{1}{u}du$$

(by substitution $u = 1+4x^2, du = 8xdx$) $= x\arctan 2x - \frac{1}{4}\ln(1+4x^2)$

so integral $\qquad I = F(\frac{1}{2}) - F(0) = \frac{1}{2}\arctan 1 - \frac{1}{4}\ln 2 = \dfrac{\pi}{8} - \frac{1}{4}\ln 2$

(since $\arctan 0 = 0, \ln 1 = 0, c = 0$).

For 5)

$$I = \int\limits_0^{\frac{1}{2}} \frac{2x+1}{4x^2+1}dx = \int\limits_0^{\frac{1}{2}} \frac{2x}{4x^2+1}dx + \int\limits_0^{\frac{1}{2}} \frac{1}{4x^2+1}dx = \tfrac{1}{2}\int\limits_0^{1} \frac{u}{u^2+1}du + \tfrac{1}{2}\int\limits_0^{1} \frac{1}{u^2+1}du.$$

by substitution $2x = u, 2dx = du$,we get

$$\int_0^{\frac{1}{2}} \frac{2x+1}{4x^2+1}\ dx = \left[\tfrac{1}{4}\ln(u^2+1) + \tfrac{1}{2}\arctan u\right]_0^1 =$$

$= \frac{1}{4}\ln 2 + \frac{1}{2}\arctan 1 = \dfrac{\ln 2}{4} + \dfrac{\pi}{8}$, since $\ln 1 = 0, \arctan 0 = 0$.

For 6)

by parts $\left(du = \sin\frac{x}{3}dx.\text{by integr...}u = -3\cos\frac{x}{3}, v = e^{3x}.\text{by diff...}dv = 3e^{3x}dx\right)$

$$F(x) = \int e^{3x}\sin\tfrac{x}{3}dx = e^{3x}(-3\cos\tfrac{x}{3}) - \int(-3)\cos\tfrac{x}{3}\cdot e^{3x}\cdot 3dx =$$

$$= -3e^{3x}\cos\tfrac{x}{3} + 9\int\cos\tfrac{x}{3}\cdot e^{3x}dx =$$

by parts again$\left(du = \cos\frac{x}{3}dx...u = 3\sin\frac{x}{3}, v \text{ ..as above}\right)$

$$= -3e^{3x}\cos\tfrac{x}{3} + 9\left[3\sin\tfrac{x}{3}\cdot e^{3x} - \int 3\sin\tfrac{x}{3}\cdot 3e^{3x}dx\right] =$$

$= -3e^{3x}\cos\frac{x}{3} + 27e^{3x}\sin\frac{x}{3} - 81F(x)$,solve for $F(x)$:

$82 \cdot F(x) = -3e^{3x}\cos\frac{x}{3} + 27e^{3x}\sin\frac{x}{3} + c$,

thus $F(x) = \frac{1}{82}(-3e^{3x}\cos\frac{x}{3} + 27e^{3x}\sin\frac{x}{3}) + c$.

For 7)

Domain $x \geq 0$ and subst $u = 2+\sqrt{x} \qquad (u-2)^2 = x \qquad 2(u-2)du = dx$

$$\int \frac{\sqrt{x}}{2+\sqrt{x}}dx. = \int \frac{u-2}{u}\cdot 2(u-2)du = 2\int \frac{u^2-4u+4}{u}du =$$

$= 2\int \left(u - 4 + \frac{4}{u}\right) du = u^2 - 8u + 8\ln|u| =$

$= \left(2+\sqrt{x}\right)^2 - 8\left(2+\sqrt{x}\right) + 8\ln\left(2+\sqrt{x}\right) + c =$

(also) $= x - 4\sqrt{x} + 8\ln(2+\sqrt{x}) + c.$

For 8)

use inverse subst. $u = 2 + \sqrt{x+1} \quad (u-2)^2 = x+1$ and $dx = 2(u-2)du$

For $x \geq -1$

$$\int \frac{1}{2+\sqrt{x+1}}dx = \int \frac{2(u-2)}{u}du = 2\int \frac{u-2}{u}du = 2\int du - 4\int \frac{1}{u}du =$$

$= 2u - 4\ln|u| + c = 4 + 2\sqrt{x+1} - 4\ln\left|2+\sqrt{x+1}\right| + c.$

OR by subst.

$u = 2 + \sqrt{x+1}\ du = \frac{1}{2\sqrt{x+1}}dx$ and

$$\int \frac{1}{2+\sqrt{x+1}} \cdot \frac{2\sqrt{x+1}}{2\sqrt{x+1}}dx = \int \frac{2\sqrt{x+1}}{2+\sqrt{x+1}} \cdot \frac{dx}{2\sqrt{x+1}} = \int \frac{2(u-2)}{u}du = ..$$

as above.

For 9)

$\int_{-1}^{0} x^2 e^{-3x}dx =$(by parts :

$du = e^{-3x}dx$...by integr.... $u = -\frac{1}{3}e^{-3x}, v = x^2$.by diff...$dv = 2xdx$)

$$= \left[\frac{e^{-3x}}{-3}x^2\right]_{-1}^{0} - \int_{-1}^{0} \frac{e^{-3x}}{-3} \cdot 2xdx = 0 + \frac{e^3}{3} + \frac{2}{3}\int_{-1}^{0} xe^{-3x}dx$$

(by parts again: $du =$ as above,$v = x$...$dv = 1dx$)

$$= \frac{e^3}{3} + \frac{2}{3}\left[\frac{e^{-3x}}{-3}x - \int \frac{e^{-3x}}{-3} \cdot 1dx\right]_{-1}^{0} = \frac{e^3}{3} + \frac{2}{3}\left[0 + \frac{-e^3}{3}\right] + \frac{2}{9}\left[\frac{e^{-3x}}{-3}\right]_{-1}^{0} =$$

$$= e^3\left(\frac{1}{3} - \frac{2}{9}\right) - \frac{2}{27}\left(1 - e^3\right) = e^3\frac{9-6+2}{27} - \frac{2}{27} = \frac{5e^3 - 2}{27}.$$

For 10)

Complete the square $x^2 - 4x = (x-2)^2 - 4$ so

$$I = \int_0^2 x\sqrt{4x - x^2}dx = \int_0^2 x\sqrt{4 - (x-2)^2}dx =$$

(using $u = x - 2, x = u + 2, du = dx, u = -2, 0$)

$$= \int_{-2}^{0} (u+2)\sqrt{4-u^2}du = \int_{-2}^{0} u\sqrt{4-u^2}du + 2\int_{-2}^{0} \sqrt{4-u^2}du =$$

for the first integral use substitution $4 - u^2 = v, udu = -\frac{1}{2}dv,$

for the second use **Table** or by parts or by subst. $u = 2\sin t, du = 2\cos tdt$

so $I = -\frac{1}{2}\int\limits_0^4 \sqrt{v}dv + \left[u\sqrt{4-u^2} + 4\arcsin\frac{u}{2}\right]_{-2}^{0}$

$\left[\text{Or} \qquad 4\int\limits_0^{\frac{\pi}{2}} \cos t\sqrt{4-4\sin^2 t}dt = (...8\cos^2 t) = 4\int\limits_0^{\frac{\pi}{2}} (1+\cos 2t)\,dt\right]$

$= \left[-\frac{1}{3}v^{\frac{3}{2}}\right]_0^4 + 4\arcsin 1 = -\frac{8}{3} + 4\cdot\frac{\pi}{2} = 2\pi - \dfrac{8}{3}.$

For 11)

by parts:

$du = (3x+2)\,dx$.by integr...$u = \frac{3}{2}x^2 + 2x, v = \ln x$..by diff.....$dv = \dfrac{dx}{x}$

$\int (3x+2)\ln x = \left(\frac{3}{2}x^2+2x\right)\ln x - \int\left(\frac{3}{2}x^2+2x\right)\cdot\frac{1}{x}dx =$

$= \left(\frac{3}{2}x^2+2x\right)\ln x - \int\left(\frac{3}{2}x+2\right)dx$

$= \left(\frac{3}{2}x^2+2x\right)\ln x - \frac{3}{4}x^2 - 2x + c \qquad \text{for } x > 0.$