MATH 253 HANDOUT #4

A

1. Find the trapezoid approximation of $\int\limits_0^2 \frac{1}{1+x^2}\,dx$ for $n = 4$.

   Can you calculate the integral exactly?

2. Find the volume of the solid obtained by rotating the region D around y-axis,where D

is the region bounded by the graph of $y = \frac{6}{x}$ and the lines $x = 2, y = 2$.

3.Find the arclength of the curve $y = \ln(\sin x)$ between $x = \frac{\pi}{4}$ and $x = \frac{\pi}{2}$.

Solutions

For 1)

The length of the subintervals is $\triangle x = h = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}$, so the points of partition are: $x_0 = 0, x_1 = \frac{1}{2}, x_2 = 1, x_3 = \frac{3}{2}, x_4 = 2$, then the values of $y = f(x) = \frac{1}{x^2+1}$ are $y_0 = 1, y_1 = \frac{1}{1+\frac{1}{4}} = \frac{4}{5}, y_2 = \frac{1}{2}, y_3 = \frac{1}{1+\frac{9}{4}} = \frac{4}{13}, y_4 = \frac{1}{5}$,

so $T_4 = \frac{1}{2}\left[\frac{1}{2} + \frac{4}{5} + \frac{1}{2} + \frac{4}{13} + \frac{1}{10}\right] = \frac{1}{2}\left[1 + \frac{9}{10} + \frac{4}{13}\right] = \frac{1}{2}\cdot\frac{130+117+40}{130} =$

$= \frac{287}{260} = 1.1038462$

By direct integration:

$$\int\limits_0^2 \frac{1}{1+x^2}\,dx = [\arctan x]_0^2 = \arctan 2 - \arctan 0 = \arctan 2 = 1.1071487,$$

so we got an approximation of $\arctan 2 \doteq 1.1038462$

For 2)

First,sketch the region :hyperbola $y = \frac{6}{x}$.... top $y = 2$... bottom

intersection at the point $x = 3, y = 2$

y-axis→" shells"→ dx

$$V = 2\pi\int\limits_2^3 RH\,dx,\text{ where } R = x,\text{and } H = \text{"}hyperbola - line\text{"} = \frac{6}{x} - 2,\text{so}$$

$$V = 2\pi\int\limits_2^3 x\cdot\left(\frac{6}{x} - 2\right)dx = 2\pi\cdot 6\,(3-2) - 2\pi\int\limits_2^3 2x\,dx =$$

$$= 12\pi - 2\pi\left[x^2\right]_2^3 = 12\pi - 2\pi\,(9-4) = 2\pi.$$

For 3)

First $\quad y' = \frac{\cos x}{\sin x}$ then $\quad 1 + (y')^2 = \frac{\sin^2 x + \cos^2 x}{\sin^2 x} = \csc^2 x$

so $\quad l = \int\limits_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{1+(y')^2}\,dx = \int\limits_{\frac{\pi}{4}}^{\frac{\pi}{2}} \csc x\,dx = \text{(Table)} = [\ln|\csc x - \cot x|]_{\frac{\pi}{4}}^{\frac{\pi}{2}} =$

$\left(\csc\frac{\pi}{2} = 1, \cot\frac{\pi}{2} = 0, \csc\frac{\pi}{4} = \sqrt{2}, \cot\frac{\pi}{4} = 1\right) = \ln 1 - \ln\left(\sqrt{2}-1\right) = \ln\left(\sqrt{2}+1\right)$

since $\frac{1}{\sqrt{2}-1} = \sqrt{2}+1$.

B

1.Find midpoint approximation of $\int\limits_{1}^{3} \frac{1}{x-4}\ dx$ for $n = 3$.

Use it to approximate $\ln 3$.

2.Find the volume of the solid obtained by rotating the triangle T with vertices at the points $(1,1), (1,-2)$ and $(2,0)$ around y-axis.

3.Find the length of the curve $y^3 = x^2$ between points $O\,(0,0)$ and $P\,(8,4)$.

Solutions

For 1)

the length of the subintervals is $\triangle x = h = \frac{b-a}{n} = \frac{3-1}{3} = \frac{2}{3}$

so partition points are: $x_0 = 1, x_1 = \frac{5}{3}, x_2 = \frac{7}{3}, x_3 = \frac{9}{3} = 3$, and midpoints are:

$m_1 = \frac{4}{3}, m_2 = \frac{6}{3} = 2, m_3 = \frac{8}{3}$,and $x - 4 = -\frac{8}{3}, -2, -\frac{4}{3}$,

and values of $f\,(x) = \frac{1}{x-4}$ at the midpoints are: $-\frac{3}{8}, -\frac{1}{2}, -\frac{3}{4}$ and

$M_3 = \frac{2}{3}\left[-\frac{3}{8} - \frac{1}{2} - \frac{3}{4}\right] = \frac{2}{3} \cdot -\frac{3+4+6}{8} = -\frac{13}{12}$.

Now,we can evaluate the integral directly:$[\ln|x-4|]_1^3 = \ln 1 - \ln 3 = -\ln 3$,

so $\ln 3 \doteq \frac{13}{12} = 1.083$

For 2)

y-axis$\rightarrow$ "shells"$\rightarrow dx$

we need two lines first,through $(1,1)$ and $(2,0) : y = 2 - x$......top

through $(1,-2)$ and $(2,0) : y = 2x - 4$.....bottom

Radius of the cylindrical shell is x,the height

$H =$ "top–bottom"$=(2-x) - (2x-4) = 6 - 3x = 3(2-x)$

so $V = 2\pi \int\limits_a^b RH dx = 2\pi \int\limits_1^2 x \cdot 3\,(2-x)\,dx = 6\pi\,[x^2]_1^2 - 2\pi\,[x^3]_1^2 =$

$= 6\pi\,[4-1] - 2\pi\,[8-1] = 4\pi.$

For 3)

First $y = x^{\frac{2}{3}}$ so $y' = \frac{2}{3}x^{-\frac{1}{3}}$ and $1 + (y')^2 = 1 + \dfrac{4}{9x^{\frac{2}{3}}} = \dfrac{9x^{\frac{2}{3}} + 4}{9x^{\frac{2}{3}}}$

and $\quad l = \int\limits_0^8 \sqrt{1+(y')^2}dx = \int\limits_0^8 \sqrt{\dfrac{9x^{\frac{2}{3}}+4}{9x^{\frac{2}{3}}}}dx = \int\limits_0^8 \dfrac{\sqrt{9x^{\frac{2}{3}}+4}}{3x^{\frac{1}{3}}}dx =$

$$\left(\begin{array}{c} \text{subst. } u = 9x^{\frac{2}{3}} + 4, du = \frac{18}{3}x^{-\frac{1}{3}}dx, \frac{1}{18}du = \dfrac{dx}{3x^{\frac{1}{3}}} \\ \text{and } u = 4 \text{ for } x = 0, u = 40 \text{ for } x = 8 \end{array}\right)$$

thus $\quad l = \frac{1}{18}\int\limits_4^{40} \sqrt{u}du = \frac{1}{18}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_4^{40} = \frac{4^{\frac{3}{2}}}{27}\left[10^{\frac{3}{2}} - 1\right] = \frac{8}{27}\left[10\sqrt{10} - 1\right].$

k

C

1. .Find midpoint approximation of $\int_1^3 \frac{1}{x}\, dx$ for $n = 4$.

Can you calculate the error?

2.Find the volume of the solid obtained by rotating the region D around x-axis, where D is in the first quadrant below the graph of $y = 2 - x^2$ and above the line $y = x$.

3.Find the length of the part of the circle $x^2 + y^2 = 5$ between points $Q\,(2, 1)$ and $P\,(1, 2)$.

Solution

For 1)

The length of the subintervals is $\triangle x = h = \frac{b-a}{n} = \frac{3-1}{4} = \frac{1}{2}$, so the points of partition are:
$x_0 = 1, x_1 = \frac{3}{2}, x_2 = 2, x_3 = \frac{5}{2}, x_4 = 3$, then midpoints are:
$m_1 = \frac{5}{4}, m_2 = \frac{7}{4}, m_3 = \frac{9}{4}, m_4 = \frac{11}{4}$,and the values of $f(x) = \frac{1}{x}$ are: $\frac{4}{5}, \frac{4}{7}, \frac{4}{9}, \frac{4}{11}$
So $M_4 = \frac{1}{2}\left[\frac{4}{5} + \frac{4}{7} + \frac{4}{9} + \frac{4}{11}\right] = 2\left[\frac{16}{55} + \frac{16}{7\cdot 9}\right] = 32\left[\frac{1}{55} + \frac{1}{63}\right] = 1.0897547$
By direct integration:

$$\int_1^3 \frac{1}{x}\, dx = [\ln x]_1^3 = \ln 3 = 1.0986123,\text{so the error is } 0.0088576$$

since error=exact value –approximation

For 2)

First,find the intersection of the parabola and line:$y = 2 - x^2 = x, x > 0$,so
$0 = x^2 + x - 2 = (x + 2)(x - 1) \qquad x = 1, y = 1$
parabola ...top=$r_{outside}$,line...bottom= r_{inside}
x-axis→ "washers"→dx

$$V = \pi \int_0^1 \left(r_{outside}^2 - r_{inside}^2\right) dx = \pi \int_0^1 \left[\left(2 - x^2\right)^2 - x^2\right] dx =$$

$$= \pi \int_0^1 \left[4 - 4x^2 + x^4 - x^2\right] dx = \pi \int_0^1 \left[4 - 5x^2 + x^4\right] dx =$$

$$= \pi \left[4x - \tfrac{5}{3}x^3 + \tfrac{1}{5}x^5\right]_0^1 = \pi\left[4 - \tfrac{5}{3} + \tfrac{1}{5}\right] = \frac{38}{15}\pi.$$

For 3)

First $y = \sqrt{5 - x^2}$ for $1 \le x \le 2$

then $y' = \dfrac{-x}{\sqrt{5 - x^2}}$ and $1 + (y')^2 = \dfrac{5 - x^2 + x^2}{5 - x^2}$

thus $$l = \int_1^2 \sqrt{1 + (y')^2} dx = \int_1^2 \frac{\sqrt{5}}{\sqrt{5 - x^2}} dx = \sqrt{5}\left[\arcsin \tfrac{2}{\sqrt{5}} - \arcsin \tfrac{1}{\sqrt{5}}\right].$$

D

1. Find the trapezoid approximation of $\int\limits_1^2 \frac{1}{x^2}\, dx$ for $n = 3$.

Can you calculate the error?

2. Find the volume of the solid obtained by rotating the region D around x-axis, where D nis the region bounded by the graph of $y = \frac{2}{x}$ and the lines $x = 1, y = 4$.

3.Find the length of the curve $y^2 = x^3$ between points O$(0, 0)$ and $P\left(4, -8\right)$.

Solution

For 1)

The length of the subintervals is $\triangle x = h = \frac{b-a}{n} = \frac{2-1}{3} = \frac{1}{3}$, so the points of partition are: $x_0 = 1, x_1 = \frac{4}{3}, x_2 = \frac{5}{3}, x_3 = 2$, then the values of $y = f(x) = \frac{1}{x^2}$

$y_0 = 1, y_1 = \frac{9}{16}, y_2 = \frac{9}{25}, y_3 = \frac{1}{4}$,

So $T_3 = \frac{1}{3}\left[\frac{1}{2} + \frac{9}{16} + \frac{9}{25} + \frac{1}{8}\right] = \frac{1}{3}\left[9 \cdot \frac{41}{400} + \frac{5}{8}\right] = \frac{123}{400} + \frac{5}{24} = \frac{619}{1200} = 0.51583$

By direct integration:

$\int\limits_1^2 \frac{1}{x^2}\, dx = \left[\frac{-1}{x}\right]_1^2 = \frac{-1}{2} + 1 = \frac{1}{2}$,so the error is 0.01583.

For 2)

First,sketch the region :hyperbola $y = \frac{2}{x}$...bottom;line $y = 4$.. top

intesection hyperbola and line $y = 4$ is at $x = \frac{1}{2}$

top=$r_{outside}$ bottom= r_{inside}

x-axis→ " washers"→ dx

$$V = \pi \int\limits_{\frac{1}{2}}^{1} \left(r_{outside}^2 - r_{inside}^2\right) dx = \pi \int\limits_{\frac{1}{2}}^{1} \left[4^2 - \left(\frac{2}{x}\right)^2\right] dx =$$

$$= 16\pi \cdot \left(1 - \tfrac{1}{2}\right) - 4\pi \int\limits_{\frac{1}{2}}^{1} x^{-2} dx = 8\pi - 4\pi \left[-\tfrac{1}{x}\right]_{\frac{1}{2}}^{1} = 8\pi + 4\pi\left(1 - 2\right) = 4\pi.$$

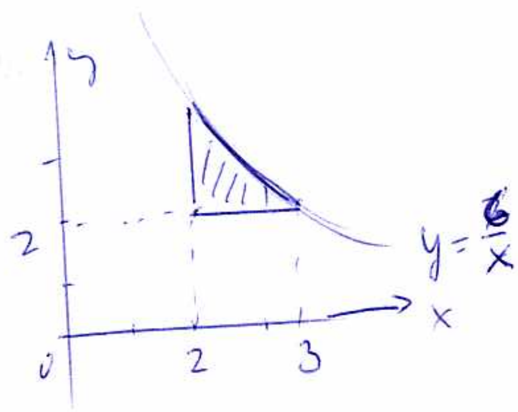
For 3)

First $y = -x^{\frac{3}{2}}$ for $0 \le x \le 4$ since y is negative

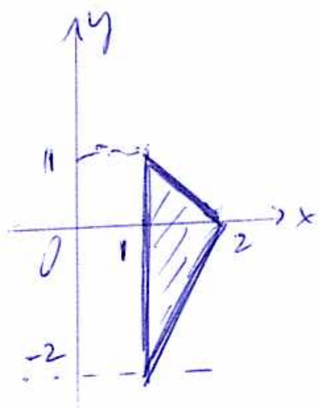
then $y' = -\frac{3}{2}x^{\frac{1}{2}}$ and $1 + (y')^2 = 1 + \frac{9}{4}x = \frac{4 + 9x}{4}$

so $l = \int\limits_0^4 \sqrt{1 + (y')^2} dx = \frac{1}{2} \int\limits_0^4 \sqrt{4 + 9x} dx$ (subst. $u = 9x + 4$) $=$

$$= \frac{1}{18} \int\limits_4^{40} \sqrt{u} du = \frac{1}{18} \left[\tfrac{2}{3}\left(u\right)^{\frac{3}{2}}\right]_0^4 = \frac{1}{27}\left[40^{\frac{3}{2}} - 4^{\frac{3}{2}}\right] = \tfrac{4^{\frac{3}{2}}}{27}\left[10^{\frac{3}{2}} - 1\right] = \tfrac{8}{27}\left[10\sqrt{10} - 1\right].$$



A_2



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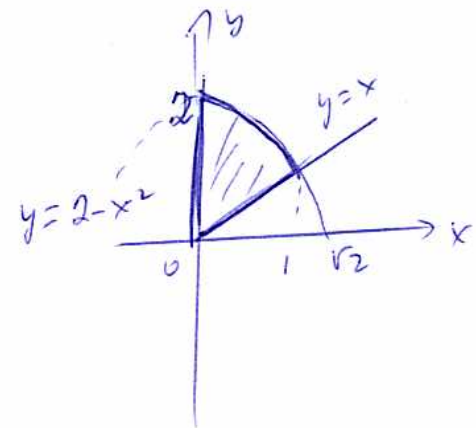
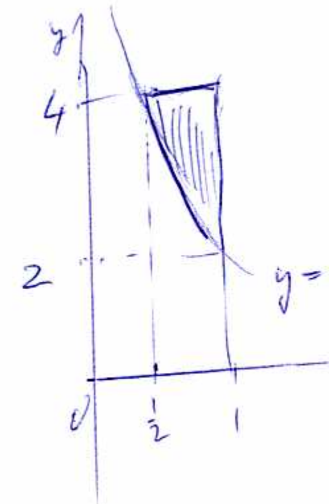


C_2



D_2