MATH 253
Handout # 5-SOLUTION

A

1. Solve $y' + y\sin x = \cos x \sin x$, $y(\dfrac{\pi}{2}) = 3$.

For 1)

the equation is linear $\quad p(x) = \sin x$,$P(x) = -\cos x$ and $\mu = e^{-\cos x}$

multiply by the ntegrating factor μ

$y'e^{-\cos x} + y\sin x e^{-\cos x} = (ye^{-\cos x})' = e^{-\cos x}\cos x \sin x$

so $ye^{-\cos x} = \displaystyle\int e^{-\cos x}\cos x \sin x \; dx + c$

by subst.$u = -\cos x, du = \sin x dx$ the integral$= -\displaystyle\int e^u u \; du = e^u - ue^u$

so

$ye^{-\cos x} = e^{-\cos x}(1 + \cos x) + c$ and multiply by $e^{\cos x}$

$y = 1 + \cos x + ce^{\cos x}$,finally for $x = \dfrac{\pi}{2} \quad 3 = 1 + 0 + c$ so $c = 2$

and the solution is $\quad y = 1 + \cos x + 2e^{\cos x}$ for any x.

2.Find the Taylor polynomial of degree 3 of $f(x) = \ln(3x-2)$ around $a = 1$, then use it to approximate $\ln 2$.

For 2)

first ....$a_0 = f(1) = \ln 1 = 0$, then

using Chain Rule :$f'(x) = \dfrac{3}{3x-2}$...................... $a_1 = f'(1) = 3$

$f''(x) = 3\,(-1)\,(3x-2)^{-2}\cdot 3 = \dfrac{-9}{(3x-2)^2}$..........$a_2 = \frac{f''(1)}{2} = \frac{-9}{2}$

$f'''(x) = -9\,(-2)\,(3x-2)^{-3}\cdot 3 = \dfrac{54}{(3x-2)^3}$.......$a_3 = \frac{f'''(1)}{3!} = \frac{54}{6} = 9$

and

$T_3(x) = 3(x-1) - \frac{9}{2}(x-1)^2 + 9(x-1)^3$

Now to get $\ln 2$ we have to find x : $\quad 3x - 2 = 2$...so $x = \frac{4}{3}$ and

$\ln 2 \doteq T_3(\frac{4}{3}) = 3\cdot\frac{1}{3} - \frac{9}{2}\cdot\frac{1}{9} + 9\cdot\frac{1}{27} = 1 - \frac{1}{2} + \frac{1}{3} = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$

$\ln 2 \doteq \frac{5}{6}$ (not a very good approximation)

3. Find the general solution, in the explicit form, of

$(x+1)y' + (1+y)x^2 = 0$.

For 3)

re-arrange the equation $\quad \frac{dy}{dx} = y' = \dfrac{-x^2(1+y)}{x+1}$.......separable

separate

$\dfrac{1}{1+y}dy = \dfrac{-x^2}{x+1}dx$ now integrate $\quad \int \frac{1}{1+y}dy = \int \dfrac{-x^2}{x+1}dx$

on the right a rational function so long division first or subst. $u = x + 1$

$\int \dfrac{1 - x^2 - 1}{x+1}dx = \int \frac{(1-x)(x+1)}{x+1}dx - \int \frac{1}{x+1}dx = x - \frac{x^2}{2} - \ln|x+1| + c$

back to equation

$\ln|1+y| = -\frac{x^2}{2} + x - \ln|x+1| + c \qquad y + 1 = \pm e^c \cdot e^{-\frac{x^2}{2}+x} \cdot e^{-\ln|x+1|}$

finally denote $A = \pm e^c$

$y = -1 + \dfrac{Ae^{-\frac{x^2}{2}+x}}{x+1}$ for $x \neq -1$

4. Find the solution,in the simpliest form, of the initial value problem

$y' = \dfrac{y}{x-y} \quad y(-2) = 1.$

For 4)

$y' = \dfrac{y}{x-y} \quad$ not sep.,not linear so re-arrange for $x \neq 0$

$y' = \dfrac{\frac{y}{x}}{1-\frac{y}{x}}$ subst.$u = \frac{y}{x}, y' = u'x + u$ gives a separable equation

$u'x + u = \dfrac{u}{1-u} \qquad u'x = \dfrac{u}{1-u} - u = \frac{u-u+u^2}{1+u} = \dfrac{u^2}{1-u}$

separate

$\dfrac{1-u}{u^2}du = \dfrac{dx}{x}$ integrate $\int \left(\frac{1}{u^2} - \frac{1}{u}\right) du = \ln|x| + c$

so

$\frac{-1}{u} - \ln|u| = \ln|x| + c \quad$ back to $y : \ln\left|\frac{y}{x}\right| = \ln|y| - \ln|x|$

thus

$\frac{-x}{y} - \ln|y| + \ln|x| = \ln|x| + c$

and cancel $\ln|x|$ and multiply by y :

$-y\ln|y| - cy = x$ for any $y \neq 0$ now $x = -2, y = 1$,solve for c

$0 - c = -2$ and together $\quad x = -y\ln|y| - 2y, y \neq 0.$

B

1. Find the quadratic approximation of $f(x) = e^{1-4x^2}$around $a = \frac{1}{2}$, then use it to approximate $e^{\frac{3}{4}}$.

For 1)

first ....$a_0 = f(\frac{1}{2}) = e^0 = 1$, then

using Chain Rule :$f'(x) = e^{1-4x^2}(-8x)$ .... $a_1 = f'(\frac{1}{2}) = -4$

Product Rule: $f''(x) = (-8)\,e^{1-4x^2}(1-8x^2)$.........$a_2 = \frac{f''(\frac{1}{2})}{2} = \frac{8}{2} = 4$

and

$T_2(x) = 1 - 4(x - \frac{1}{2}) + 4(x - \frac{1}{2})^2$

Now to estimate $e^{\frac{3}{4}}$ we have to find x such that $1 - 4x^2 = \frac{3}{4}$,

so $\frac{1}{4} = 4x^2$,and $x^2 = \frac{1}{4^2}, x = \pm\frac{1}{4}$ but $\frac{1}{4}$ is closer to $\frac{1}{2}$

$e^{\frac{3}{4}} \doteq T_2(\frac{1}{4}) = 1 - 4 \cdot \frac{-1}{4} + 4 \cdot \frac{1}{16} = 2.25$ (not a very good approximation)

2.Find the general solution of $y' - 2xy = x$.

For 2)

the equation is linear $\qquad p(x) = -2x$,$P(x) = -x^2$ and $\mu = e^{-x^2}$

multiply by the integrating factor μ

$y'e^{-x^2} - 2xye^{-x^2} = \left(ye^{-x^2}\right)' = xe^{-x^2}$ so $ye^{-x^2} = \displaystyle\int xe^{-x^2}dx + c$

by subst. $u = -x^2, du = -2xdx$,integral $= -\dfrac{1}{2}\displaystyle\int e^u du = \dfrac{-1}{2}e^u$

so $\qquad ye^{-x^2} = -\dfrac{1}{2}e^{-x^2} + c$ and multiply by $e^{x^2}to$ get

$y = \dfrac{-1}{2} + ce^{x^2}$ for any x.

Note: The equation is also separable.

3. Find the general solution, in the explicit form, of
$(x^2+1)y' + 2(1+y)x^2 = 0$.

For 3)

re-arrange the equation $\frac{dy}{dx} = y' = \dfrac{-2x^2(1+y)}{x^2+1}$.......sep.

$\frac{1}{1+y}dy = \dfrac{-2x^2}{x^2+1}dx$ now integrate $\int \frac{1}{1+y}dy = \int \dfrac{-2x^2}{x^2+1}dx$

$\ln|1+y| = -2\int \dfrac{x^2+1-1}{x^2+1}dx = -2x + 2\int \frac{1}{x^2+1}dx = -2x + 2\arctan x + c$

$|1+y| = e^{-2x+\arctan x} \cdot e^c \qquad y = -1 + Ae^{-2x+\arctan x}$, where $A = \pm e^c$

4. Find the general solution,in the simpliest form, of $y' = \dfrac{y}{x+y}$.

For 4)

$y' = \dfrac{y}{x+y}$ not sep.,not linear so re-arrange

$y' = \dfrac{\frac{y}{x}}{1+\frac{y}{x}}$ subst.$u = \frac{y}{x}, y' = u'x + u$ gives a separable equation

$u'x + u = \dfrac{u}{1+u} \qquad u'x = \dfrac{u}{1+u} - u = \frac{u-u-u^2}{1+u} = -\dfrac{u^2}{1+u}$

separate

$\dfrac{1+u}{u^2}du = -\frac{dx}{x}$ integrate $\int \left(\frac{1}{u^2} + \frac{1}{u}\right) du = -\ln|x| + c$

so

$\frac{-1}{u} + \ln|u| = -\ln|x| + c$ back to y :using $\ln\left|\frac{y}{x}\right| = \ln|y| - \ln|x|$ we get

$\frac{-x}{y} + \ln|y| - \ln|x| = -\ln|x| + c$ and $y\ln|y| - cy = x$ for any $y \neq 0$

C

1..Find the Taylor polynomial of degree 3 of $f(x) = \arctan(3x)$ around $a = 0$, then use it to approximate $\pi = 4\arctan 1$.

For 1)

first ....$a_0 = f(0) = \arctan 0 = 0$, then

using Chain Rule :$f'(x) = \dfrac{3}{1+(3x)^2} = \dfrac{3}{1+9x^2}$........ $a_1 = f'(0) = 3$

$f''(x) = 3\,(-1)\,(1+9x^2)^{-2} \cdot 18x = \dfrac{-54x}{(1+9x^2)^2}$..........$a_2 = \frac{f''(0)}{2} = 0$

by Q.R.

$f'''(x) = (-54)\,\dfrac{(1+9x^2)^2 - x \cdot 2\,(1+9x^2)\,18x}{(1+9x^2)^4}$..........$a_3 = \frac{f'''(0)}{3!} = \frac{-54}{6} = -9$

and $T_3(x) = 3x - 9x^3$

Now to get $\arctan 1$ we have to substitute for $x = \frac{1}{3}$

$\arctan 1 \doteq T_3(\frac{1}{3}) = 3 \cdot \frac{1}{3} - 9 \cdot \frac{1}{27} = 1 - \frac{1}{3} = \frac{2}{3}$ $\pi = 4\arctan 1 \doteq \frac{8}{3}$

(not a very good approximation)

2. Find the general solution of $xy' = x^2 + y$.

For 2)

for $x \neq 0$ we can re-write the equation as $y' - \dfrac{y}{x} = x$

so it is linear and $p(x)=\dfrac{-1}{x}$,$P(x)=-\ln|x|=\ln\left|\dfrac{1}{x}\right|$ and $\mu=\dfrac{1}{x}$
multiply by the integrating factor μ
$y'\dfrac{1}{x}-\dfrac{y}{x^2}=\left(\dfrac{y}{x}\right)'=1$ and $\dfrac{y}{x}=\int dx+c=x+c$
and finally $\quad y=x^2+cx$ for any x.
3. Find the general solution, in the simpliest form,
$(x+y)y'+y-3x=0.$
For 3)
re-arrange the equation $\quad y'=\dfrac{3x-y}{x+y}$.......not sep.,non-linear but homog.
$y'=\dfrac{3x-y}{x+y}=\dfrac{3-\frac{y}{x}}{1+\frac{y}{x}}$.Now substitution.....$u=\frac{y}{x}, y'=u'x+u$ gives
$u'x+u=\dfrac{3-u}{1+u}$........$u'x=\dfrac{3-u}{1+u}-u=\dfrac{3-u-u-u^2}{1+u}=\dfrac{3-2u-u^2}{1+u}$......separable
$\dfrac{1+u}{3-2u-u^2}du=\dfrac{dx}{x}$ for integration of the left-hand side use partial fractions
$\dfrac{1+u}{3-2u-u^2}=-\dfrac{1+u}{u^2+2u-3}=-\dfrac{1+u}{(u-1)(u+3)}=\dfrac{A}{u-1}+\dfrac{B}{u+3}$
so $\;-(1+u)=A(u+3)+B(u-1)$.......$u=1$......$-2=4A$.....$A=\frac{-1}{2}$
for $u=-3$.........$2=-4B$......$B=\frac{-1}{2}$....back to the diff.equation

$\frac{-1}{2}\ln|(u+3)(u-1)|=\ln|x|+c$..... $\ln|(u+3)(u-1)|=-2\ln|x|+C$
apply exp.function to both sides
$|(u+3)(u-1)|=e^C\cdot|x|^{-2}$ .... $(u+3)(u-1)=A\frac{1}{x^2}$....back to y
$\left(\frac{y+3x}{x}\right)\left(\frac{y-x}{x}\right)=\frac{A}{x^2}$..... $(y+3x)(y-x)=A$......general solution
4. Find the explicit solution of the initial value problem
$x\sin y+y'(x^2+1)\cos y=0, y(0)=-\frac{\pi}{2}.$
For 4)
$y'(x^2+1)\cos y=-x\sin y$ separable....$\dfrac{\cos y}{\sin y}dy=\dfrac{-x}{x^2+1}dx$
integrate
$\int\frac{\cos y}{\sin y}dy=(\text{subst.}u=\sin y, du=\cos y dy)=\int\frac{du}{u}=\ln|u|=\ln|\sin y|$
and $\displaystyle\int\frac{-xdx}{x^2+1}=(\text{subst.}x^2+1=u, xdx=\tfrac{1}{2}du)=-\tfrac{1}{2}\ln|x^2+1|+c$
back to diff. equation
$\ln|\sin y|=\ln|x^2+1|^{\frac{-1}{2}}+c$................ $\sin y=\pm e^c\cdot(x^2+1)^{\frac{-1}{2}}=\dfrac{A}{\sqrt{x^2+1}}$
to find A substitute $x=0, y=-\frac{\pi}{2}$
$\sin\frac{-\pi}{2}=-1=A\cdot 1$ so $A=-1$ finally $\sin y=-\dfrac{1}{\sqrt{x^2+1}}$
and $\;y=\arcsin\frac{-1}{\sqrt{x^2+1}}$ for any x since $-1\leq\frac{-1}{\sqrt{x^2+1}}<0$
D
1.Solve $y'-y=e^x\ln x$, $y(1)=-1.$
For 1)
it is linear and $p(x)=-1$,$P(x)=-x$ and $\mu=e^{-x}$
multiply by the integrating factor μ

$y'e^{-x} - ye^{-x} = (ye^{-x})' = e^x \ln x \; e^{-x} = \ln x$ so for $x > 0$
$ye^{-x} = \int \ln x \; dx + c = x \ln x - x + c$,multiply by $e^x$to get
$y = xe^x \ln x + ce^x$ for $x > 0$.
for 2)
T_3 for $f(x) = e^{1-x^2}$ and $a = -1$
first ....$a_0 = f(-1) = e^0 = 1$, then
using Chain Rule :$f'(x) = e^{1-x^2}(-2x)$........ $a_1 = f'(-1) = 2$
by Product Rule
$f''(x) = (-2)\, e^{1-x^2}\, [1 - 2x^2]$..........$a_2 = \frac{f''(-1)}{2} = \frac{2}{2} = 1$
$f'''(x) = (-2)\, e^{1-x^2}\, [-4x - 2x + 4x^3]$..........$a_3 = \frac{f'''(-1)}{3!} = \frac{-4}{6} = \frac{-2}{3}$
and
$T_3(x) = 1 + 2(x+1) + (x+1)^2 - \frac{2}{3}(x+1)^3$

Now to estimate $e^{\frac{3}{4}}$we have to find x such that $1 - x^2 = \frac{3}{4}$
so $x^2 = \frac{1}{4}$ and $x = \pm\frac{1}{2}$ but $x = \frac{-1}{2}$ is *closer* to -1)
$e^{\frac{3}{4}} \doteq T_3(-\frac{1}{2}) = 1 + 2 \cdot \frac{1}{2} + \frac{1}{4} - \frac{2}{3} \cdot \frac{1}{8} = 2 + \frac{3-1}{12} = 2 + \frac{1}{6} = 2.16667$
3.Solve the initial value problem $yy' = 2y - x, y(1) = 0$.Can you find a solution satisfying $y(0) = 0$?
For 3)
We can rewrite the equation as $y' = 2 - \dfrac{x}{y}$ so it is clearly homog.II type,first order,
the substitution $u = \frac{y}{x}$ will give us a separable equation $u'x + u = 2 - \frac{1}{u}$
$u'x = 2 - u - \frac{1}{u} = \dfrac{2u - u^2 - 1}{u}$,separate $\dfrac{-u}{u^2 - 2u + 1}du = \dfrac{dx}{x}, \dfrac{-u}{(u-1)^2}du = \dfrac{dx}{x}$
Now $\displaystyle\int \frac{-u}{(u-1)^2}du = \int \frac{1 - u - 1}{(u-1)^2}du = \int \tfrac{-1}{u-1}du - \textstyle\int (u-1)^{-2}\, du =$
(or subst. $v = u - 1$)
$=- \ln|u-1| + (u-1)^{-1}$
so $-\ln|u-1| + (u-1)^{-1} = \ln|x| + C,$
back to $y : \ln|u-1|\,|x| = \ln|u-1| + \ln|x|$
$\left(\frac{y}{x} - 1\right)^{-1} = \ln|x| \cdot \left|\frac{y}{x} - 1\right| + C,$
$\frac{x}{y-x} = \ln|y - x| + C$ for $y \neq x$
but also $y = x$ is a solution satisfying $y(0) = 0$.
Now,if $x = 1, y = 0$,so solve for $C : -1 = C$,and the solution is
$\dfrac{x}{y-x} = \ln|y - x| - 1.$
4. Find the general solution of $x \ln x \cdot y' = y$.
For 4)
the equation is first order,separable,also linear homog.: for $x > 0, x \neq 1$
we can separate : $\dfrac{dy}{y} = \dfrac{dx}{x \ln x}$ and by integrating we get: $\ln|y| = \int \dfrac{dx}{x \ln x} =$
(by subst.$v = \ln x, dv = \frac{dx}{x}$) $= \ln|\ln x| + C,$
$|y| = e^{\ln|\ln x| + C} = e^C \cdot |\ln x|$,so $y = K \ln x, x > 0.$