MATH 253
MIDTERM HANDOUT-solution

1. Which of the following are partial fractions? Answer **YES** or **NO**.

   (a)$\dfrac{2}{3-x}$YES (b)$\dfrac{2x+1}{x^3+8}$NO since x^3

   (c)$\dfrac{3x+1}{x^2-4x+5}$YES since No real roots in denom.$D = 16-20 = -4 < 0$

   (d)$\dfrac{3x+1}{x^2-4x+3}$NO since 2 real roots in denom.$D = 16-12 = 4 > 0$

   (e)$\dfrac{3x+1}{(x^2-4x+5)^2}$YES since complex double roots in denom.

   (f)$\dfrac{x^2}{x^2+4}$NO since x^2on the top.

2. Find the inverse function f^{-1}, its domain and range if $f(x) = \arcsin(2x+3)$.

   SOLUTION

   From the properties of arcsin $\quad -1 \le 2x+3 \le 1 \quad -4 \le 2x \le -2$

   so $D_f = [-2,-1]$ and $R_f = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

   solve $y = \arcsin(2x+3)$ so $\sin y = 2x+3$ and $x = \frac{1}{2}(\sin y - 3)$

   and finally

   $f^{-1}(x) = \frac{1}{2}\sin x - \frac{3}{2}$ BUT ONLY for $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] = D_{f^{-1}}$ and $R_{f^{-1}} = [-2,-1]$.

3. Find the domain and antiderivative of $\quad f(x) = \dfrac{\ln(3x)}{x^2}$

   SOLUTION

   for domain $3x > 0$ so $D = (0,\infty)$

   now we can use by parts,integrating x^{-2} first, then diff. $\ln 3x$

   $$\int x^{-2}\ln(3x)dx = -x^{-1}\ln(3x) + \int x^{-1}\cdot\frac{1}{3x}\cdot 3dx = -\frac{\ln 3x}{x} + \int x^{-2}dx = -\frac{\ln 3x}{x} - \frac{1}{x} + c, \quad x > 0$$

   OR by or inverse subst.

   $u = \ln 3x = \ln 3 + \ln x \quad du = \dfrac{dx}{x} \quad e^u = 3x$ so $\dfrac{1}{x} = \dfrac{3}{e^u} = 3e^{-u}$

   and

   $$\int \frac{\ln 3x}{x^2}dx = \int \frac{\ln 3x}{x}\cdot\frac{dx}{x} = 3\int ue^{-u}du = \text{(by parts)}=$$

   $$= -3ue^{-u} + 3\int e^{-u}\cdot 1du = -3ue^{-u} - e^{-u} + c = \text{(back to x)}...$$

4. Find the domain and antiderivative of $f(x) = \dfrac{5x^2+2}{x^3-2x^2+x}$.

   SOLUTION

   investigate the polynomial in the denominator

   $Q(x) = x^3 - 2x^2 + x = x\left(x^2-2x+1\right) = x(x-1)^2$

   so the domain is $x \ne 0, 1$ and we can split the integrand function f

into partial fractions

$$f(x) = \frac{5x^2+2}{x^3-2x^2+x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2},$$

multiply by common denom.: $x(x-1)^2$

$(*) \quad 5x^2 + 2 = A(x-1)^2 + Bx(x-1) + Cx$, substitute $x = 0, 1, -1$

$2 = A; \quad 7 = C$ and $\quad 7 = 4A + 2B - C$ so $2B = 14 - 8 = 6, B = 3$

Now integrate

$$\int \frac{5x^2+2}{x^3-2x^2+x} dx = 2\int \frac{dx}{x} + 3\int \frac{dx}{x-1} + 7\int (x-1)^{-2}\, dx =$$

$$= 2\ln|x| + 3\ln|x-1| - \frac{7}{x-1} + c = \ln x^2 |x-1|^3 - \frac{7}{x-1} + c, \text{ for } x \neq 0, 1$$

5. (a) Is the integral $\int_4^\infty \frac{dx}{x^2-9}$ convergent or divergent?
   If convergent, evaluate it.

   (b) Is the integral $\int_0^3 \frac{dx}{x^2-9}$ convergent or divergent?
   If convergent, evaluate it.

SOLUTION

for both parts we need

$$F(x) = \int \frac{dx}{x^2-9} \text{ (by Table or Partial fraction)}$$

$$= \tfrac{1}{6}\int \left[\frac{1}{x-3} - \frac{1}{x+3}\right] dx = \tfrac{1}{6}\ln\left|\frac{x-3}{x+3}\right|$$

for a)

$$\int_4^\infty \frac{dx}{x^2-9} = \lim_{x\to\infty} F(x) - F(4) =$$

$$= \tfrac{1}{6}\ln\left|\lim_{x\to\infty} \tfrac{x-3}{x+3}\right| - \tfrac{1}{6}\ln\tfrac{1}{7} = \tfrac{1}{6}\left[\ln 1 - \ln\tfrac{1}{7}\right] = \tfrac{1}{6}\ln 7 \qquad \text{convergent.}$$

for b)

$$\int_0^3 \frac{dx}{x^2-9} = \lim_{x\to 3^-} F(x) - F(0) = \tfrac{1}{6}\lim_{x\to 3^-} \ln|x-3| - \tfrac{1}{6}\ln 6 - \tfrac{1}{6}\ln 1 = -\infty$$

since "$\ln 0+$" $= -\infty$, so the integral is divergent.

6. For $\int_0^1 \frac{3}{2+\sqrt{3x+1}} dx$

SOLUTION

use inverse subst. $\qquad u = 2 + \sqrt{3x+1} \qquad u - 2 = \sqrt{3x+1}$

$(u-2)^2 = 3x+1, \qquad \tfrac{1}{3}\left[(u-2)^2 - 1\right] = x \qquad \tfrac{2}{3}(u-2)\, du = dx,$

x	u
0	3
1	4

$$\int_0^1 \frac{3}{2+\sqrt{3x+1}} dx = 2\int_3^4 \frac{1}{u}\cdot(u-2)\, du = 2\int_3^4 du - 4\int_3^4 \frac{du}{u} = 2 - 4\ln\tfrac{4}{3}.$$