FINAL Handout
MATH 253 SOLUTION

1. **For 1a)**

first,find an antiderivative ,by parts $\int xe^x\, dx = xe^x - e^x = F(x)$

$$\int\limits_{-\infty}^{0} xe^x\, dx = F(0) - \lim_{x\to-\infty} F(x) = -1 \qquad \text{convergent}$$

since $F(0) = -1$ and $\lim\limits_{x\to-\infty} e^x = 0$,

$$\lim_{x\to-\infty} xe^x = \lim_{x\to-\infty} \frac{x}{e^{-x}} \text{"}\tfrac{-\infty}{\infty}\text{" L'H.R.} = \lim_{x\to-\infty} \frac{1}{-e^{-x}} = \text{"}\tfrac{1}{-\infty}\text{"} = 0$$

for b)

$$\int \ln x\, dx = x\ln x - x = F(x)$$

$$\int\limits_{0}^{e} \ln x\, dx = F(e) - \lim_{x\to0^+} F(x) = 0 \qquad \text{convergent}$$

since $F(e) = e\ln e - e = 0$ and

$$\lim_{x\to0^+} F(x) = \lim_{x\to0^+} x\ln x = \lim_{x\to0^+} \frac{\ln x}{x^{-1}} = \text{"}\tfrac{-\infty}{\infty}\text{" (L'H.R.)} = \lim_{x\to0^+} \frac{x^{-1}}{-x^{-2}} = \lim_{x\to0^+} (-x) = 0$$

For 2)

for $f(x) = \dfrac{x^3}{x+1}$ is defined for $x \neq -1$ and

$$f'(x) = \frac{3x^2(x+1) - x^3}{(x+1)^2} = \frac{2x^3 + 3x^2}{(x+1)^2} = \frac{2x^2}{(x+1)^2}\left(x + \tfrac{3}{2}\right) < 0$$

for $x \leq -3 < -\frac{3}{2}$

thus f is decreasing thus $one - to - one$ and continuous,

so the inverse exists on $D_f = R_{f^{-1}} = (-\infty, -3]$

now the range of f : $\lim\limits_{x\to-\infty} f(x) = \text{"}\frac{-\infty}{-\infty}\text{"} = \text{(by L'H.R.)} = \lim\limits_{x\to-\infty} \frac{3x^2}{1} = +\infty$ and

$f(-3) = \frac{-27}{-2} = \frac{27}{2}$ so $R_f = D_{f^{-1}} = \left[\frac{27}{2}, +\infty\right)$

For 3a)

we can get a sphere by rotating the region = half of a circular disk

$x^2 + y^2 \leq R^2, y > 0$ around the x-axis

so $y = \sqrt{R^2 - x^2}, x \in [-R, R]$ and using slices

$$V = \pi \int\limits_{-R}^{R} \left(\sqrt{R^2 - x^2}\right)^2 dx = \pi \int\limits_{-R}^{R} (R^2 - x^2)\, dx = 2\pi \int\limits_{0}^{R} (R^2 - x^2)\, dx = 2\pi \left[R^3 - \tfrac{R^3}{3}\right] =$$

$\frac{4}{3}\pi R^3$.

for b)

we can get a cone by rotating a triangle with vertices $(0,0)\,,(H,R)\,,(H,0)$

around x- axis slices

the region is below the line $y=\dfrac{R}{H}x$ for $x\in[0,H]$

so $V=\pi\displaystyle\int_0^H\left(\frac{R}{H}x\right)^2dx=\pi\frac{R^2}{H^2}\left[\tfrac{x^3}{3}\right]_0^H=\frac{\pi}{3}R^2H$

OR around $y-$ *axis* but the region is *below* the line $y=\frac{H}{R}\left(R-x\right)$, $x\in[0,R]$

so shells

$$V=2\pi\int_0^R x\tfrac{H}{R}\left(R-x\right)dx=2\pi\tfrac{H}{R}\int_0^R\left(Rx-x^2\right)dx=2\pi\left[H\tfrac{x^2}{2}-\tfrac{H}{R}\tfrac{x^3}{3}\right]_0^R=$$

$=\pi H\left(R^2-\frac{2}{3}R^2\right)=\frac{\pi}{3}R^2H.$

For 4).

we can calculate the arclength of the top half of a circle first and then $c=2c^+$

for $y=\sqrt{R^2-x^2},x\in[-R,R]$ we can use the formula $c^+=\displaystyle\int_{-R}^R\sqrt{1+\left(y'\right)^2}dx$

where $y'=\dfrac{-2x}{2\sqrt{R^2-x^2}}$ $\qquad 1+(y')^2=\dfrac{R^2}{R^2-x^2}$ thus

$$c=2c^+=2R\int_{-R}^R\frac{dx}{\sqrt{R^2-x^2}}=2R\left[\arcsin\tfrac{x}{R}\right]_{-R}^R=2R\left[\arcsin 1-\arcsin(-1)\right]=$$

$=2r\left[\frac{\pi}{2}+\frac{\pi}{2}\right]=2R\pi$

For 5)

for the domain solve $\quad 9-x^2\geq 0\quad 9\geq x^2\quad 3\geq|x|$ so $x\in[-3,3]=D_f$

OR

$(3-x)\left(3+x\right)\geq 0$ testing$--^{neg}-_{-3}--^{pos}---_{3}--^{neg}--$

to find $F(x)$ we can use by parts or subst.

$$F(x)=\int\sqrt{9-x^2}\cdot 1dx=x\sqrt{9-x^2}-\int x\cdot\tfrac{-2x}{2\sqrt{9-x^2}}dx=x\sqrt{9-x^2}-\int\frac{(9-x^2)-9}{\sqrt{9-x^2}}dx=$$

(split the top)=

$$=x\sqrt{9-x^2}-\int\tfrac{9-x^2}{\sqrt{9-x^2}}dx+9\int\tfrac{1}{\sqrt{9-x^2}}dx=x\sqrt{9-x^2}-F(x)+9\arcsin\tfrac{x}{3}+c$$

(derivative of arcsin)

so now we'v got an equation for unknown function $F(x)$:

$2F(x) = x\sqrt{9-x^2} + 9\arcsin\frac{x}{3} + c$ thus $\quad F(x) = \frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\arcsin\frac{x}{3} + C$

OR by inverse subst.

$x = 3\sin t \quad dx = 3\cos t \qquad$ for $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\sqrt{9-x^2} = 3\cos t$ and $\qquad t = \arcsin\frac{x}{3}$

so

$\displaystyle\int \sqrt{9-x^2}dx \;=\; 3\int \sqrt{9\left(1-\sin^2 t\right)}\cos t dt \;=\; 9\int \cos^2 t dt$ =(double angle formula)

$9\int \dfrac{1+\cos 2t}{2} dt =$

$= \frac{9}{2}\int dt + \frac{9}{2}\int \cos 2t dt = \frac{9}{2}t + \frac{9}{2}\cdot\frac{\sin 2t}{2} + c = \frac{9}{2}t + \frac{9}{2}\sin t\cos t + c$(back to x)

$= \frac{9}{2}\arcsin\frac{x}{3} + \frac{9}{2}\cdot\frac{x}{3}\frac{\sqrt{9-x^2}}{3} + c =$ ....as above.

For general circle $x^2+y^2=R^2 \qquad$ area is four times of the area below

$y = \sqrt{R^2-x^2} \qquad x \in [0, R] \qquad A = 4\displaystyle\int_0^R \sqrt{R^2-x^2}dx =$ (as above) $=$

$= 4\left[\frac{x}{2}\sqrt{R^2-x^2} + \frac{R^2}{2}\arcsin\frac{x}{R}\right]_0^R = 0 + 2R^2\arcsin 1 = \pi R^2.$

For 6)

$f(x) = \arcsin x$ so $\qquad a_0 = f(0) = 0$

$f'(x) = \frac{1}{\sqrt{1-x^2}}$ so $\qquad a_1 = f'(0) = 1$

$f''(x) = -\frac{1}{2}\left(1-x^2\right)^{-\frac{3}{2}}(-2x) = x\left(1-x^2\right)^{-\frac{3}{2}}$ so $a_2 = 0$

$f'''(x) =$ (P.R.)$=\left(1-x^2\right)^{-\frac{3}{2}} + x\left(-\frac{3}{2}\right)\left(1-x^2\right)^{-\frac{5}{2}}(-2x)$

so $\qquad a_3 = \frac{f'''(0)}{3!} = \frac{1}{6}$

and $T_3(x) = x + \frac{1}{6}x^3 \stackrel{\circ}{=} \arcsin x$ when x is close to 0

thus $\arcsin\frac{1}{3} \stackrel{\circ}{=} T_3(\frac{1}{3}) = \frac{1}{3} + \frac{1}{6\cdot 27} = \dfrac{55}{162}.$

For 7).

first order linear but first re-write:

for $x > 0 \quad y' - \frac{4}{x^2}y = x\ln x\cdot e^{-\frac{4}{x}} \qquad$ so $p(x) = -\frac{4}{x^2} \quad$ and $\qquad P(x) = \int p(x)dx = \frac{4}{x}$

and $\mu = e^{P(x)} = e^{\frac{4}{x}}$

multiply by the integrating factor the re-written equation

$e^{\frac{4}{x}}\cdot y' - \frac{4}{x^2}e^{\frac{4}{x}}y = x\ln x \quad$ check whether the left-hand side $=(ye^{\frac{4}{x}})'$

then $\qquad (ye^{\frac{4}{x}})' = x\ln x \qquad$ integrate $\quad ye^{\frac{4}{x}} = \int x\ln x dx + c$

(by parts)$\int x\ln x dx = \frac{x^2}{2}\ln x - \int\frac{x^2}{2}\cdot\frac{1}{x}dx = \frac{x^2}{2}\ln x - \frac{1}{2}\int x dx =$

$= \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + c \qquad$ so $ye^{\frac{4}{x}} = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + c$

and finally solve for $y \qquad y = e^{-\frac{4}{x}}\left(\frac{1}{2}x^2\ln x - \frac{1}{4}x^2\right) + ce^{-\frac{4}{x}} \qquad$ for $x \in\,]0, +\infty[\,.$

For 8).

second order ,linear,non-homog.,constant coeff.

first the corresponding homog. equation

$y''+6y'+9y=0$.......$r^2+6r+9=(r+3)^2=0$...$r_1=r_2=-3$ $\quad y_1=e^{-3x}, y_2=xe^{-3x}$

and general sol for homog. $\quad y_h=C_1e^{-3x}+C_2xe^{-3x}$

particular solution corresp.to the given $f(x)=18x^2$could be found in the form

$y_p=ax^2+bx+c$

$y_p'=2ax+b$ and $y_p''=2a$.......back to the equation

$L(y_p)=2a+12ax+6b+9ax^2+9bx+9c=9ax^2+(12a+9b)x+9c+2a+6b=18x^2+0x+0$

compare coefficients of x^2 : $\quad 9a=18 \quad a=2$

of x^1 : $\quad (12a+9b)=0 \quad b=-\frac{4}{3}a \quad b=-\frac{8}{3}$and finally of x^0 : $\quad 9c+2a+6b=0 \quad c=\frac{1}{9}(-6b-2a) \quad c=\frac{4}{3}$..back to $\quad y_p=2x^2-\frac{8}{3}x+\frac{4}{3}$

Together $\quad y=C_1e^{-3x}+C_2xe^{-3x}+2x^2-\frac{8}{3}x+\frac{4}{3}$ general solution,

finally the conditions $x=0, y=\frac{1}{3} \quad \frac{1}{3}=C_1+\frac{4}{3}\rightarrow C_1=-1$

differentiate $\quad y'=-3C_1e^{-3x}+C_2(e^{-3x}-3xe^{-3x})+4x-\frac{8}{3}$

$x=0, y'=3 \quad 3=3+C_2-\frac{8}{3}\rightarrow C_2=\frac{8}{3}$

Therefore the solution of the intial-value problem is

$y=-e^{-3x}+\frac{8}{3}xe^{-2x}+2x^2-\frac{8}{3}x+\frac{4}{3}$.

For 9).

second order,linear,non-homog

first,homog

$y''+9y=0$ constant coeff.,so solve $\quad r^2+9=0, r_{1,2}=\pm 3i \quad y_1=\sin 3x, y_2=\cos 3x$

general sol of homog.equation is $\quad y_h=c_1\sin 3x+c_2\cos 3x$

now

$f(x)=f_1(x)+f_2(x)$ where $f_1(x)=10\sin 2x$ and $f_2(x)=e^{-x}$

so particular solutions could be found in the form
$y_{p_1}=a\sin 2x+b\cos 2x$ and $y_{p_2}=Ae^{-x}$
now $\quad y_{p_2}'=-Ae^{-x}$ and $y_{p_2}''=Ae^{-x}$
$L(y_{p_2})=10Ae^{-x}=f_2(x)=e^{-x}$ so $A=\frac{1}{10}$ and $y_{p_2}=\frac{1}{10}e^{-x}$;
similarly,
$y_{p_1}'=2a\cos 2x-2b\sin 2x$ and $y_{p_1}''=-4a\sin 2x-2b\cos 2x$
$L(y_{p_1})=-4a\sin 2x-2b\cos 2x+9a\sin 2x+9b\cos 2x=f_2(x)=10\sin 2x$
compare the coefficients of sin and cos:
$5a\sin 2x+5b\cos 2x=10\sin 2x+0\cos 2x$
$5a=10$ and $5b=0$ so $a=2$ and $b=0$ and $y_{p_1}=2\sin 2x$.
Together
$y=y_h+y_{p_1}+y_{p_2}=c_1\sin 3x+c_2\cos 3x+2\sin 2x+\frac{1}{10}e^{-x}$.

2. **For 10 a).** $x \arcsin(2x)$

for domain solve $-1 \le 2x \le 1$ so $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ and use subst. $2x = u$ and by parts for

$$\int x \arcsin(2x)\, dx = \tfrac{1}{4}\int u \arcsin u \; du = \tfrac{1}{4}\left(\frac{u^2}{2}\arcsin u\right) - \tfrac{1}{8}\int \frac{u^2}{\sqrt{1-u^2}} du \,(\text{Table})$$

$= \frac{1}{8}u^2 \arcsin u + \frac{u}{16}\sqrt{1-u^2} - \frac{1}{16}\arcsin u + c = \frac{1}{2}x^2 \arcsin 2x + \frac{x}{8}\sqrt{1-4x^2} - \frac{1}{16}\arcsin 2x + \text{c}$

b. $$\frac{x^2+2}{x-x^2} = -\frac{x^2+2}{x^2-x} = -\frac{(x^2-x)+x+2}{x^2-x} = -1 - \frac{x+2}{x^2-x} = -1 - \frac{x+2}{x(x-1)}$$

so domain is any $x \neq 0, 1$ by Partial Fractions

$$\frac{x+2}{x(x-1)} = \frac{A}{x} + \frac{B}{(x-1)} \qquad (*) \qquad x+2 = A(x-1) + Bx \text{ so } x = 0 \text{ gives } A = -2$$

and $x = 1$ gives $B = 3$ then

$$\int \frac{x^2+2}{x-x^2} dx = \int \left[-1 - \frac{x+2}{x(x-1)}\right] dx = -x + 2\ln|x| - 3\ln|x-1| + c$$

c. $x \ln(2x+3)$

for domain $\quad 2x+3 > 0 \quad x > -\frac{3}{2}$

use subst. $u = 2x+3, du = 2dx, x = \frac{1}{2}(u-3)$ then by parts

$$\int x \ln(2x+3)\, dx = \tfrac{1}{4}\int (u-3) \ln u \; du =$$

$$= \tfrac{1}{4}\left(\tfrac{1}{2}u^2 - 3u\right)\ln u - \tfrac{1}{4}\int \left(\tfrac{1}{2}u^2 - 3u\right)\tfrac{1}{u}du =$$

$= \left(\frac{1}{8}u^2 - \frac{3}{4}u\right)\ln u - \frac{1}{16}u^2 + \frac{3}{4}u + c =$ (back to x)

$= \left[\frac{1}{8}(2x+3)^2 - \frac{3}{4}(2x+3)\right]\ln(2x+3) - \frac{1}{16}(2x+3)^2 + \frac{3}{4}(2x+3) + c$

d. $\dfrac{1}{\sqrt{e^x+1}} \qquad e^x > 0$ always so domain any x

use inverse subst. $u = \sqrt{e^x+1}, du = \frac{1}{2\sqrt{e^x+1}}e^x dx$ OR

$u^2 - 1 = e^x, \ln(u^2-1) = x, dx = 2u\frac{du}{u^2-1}$ so

$$\int \frac{1}{\sqrt{e^x+1}} dx = \int \frac{1}{e^x} \cdot \frac{e^x}{e^x\sqrt{e^x+1}} dx = \int \frac{1}{u^2-1} 2du \text{ (Part.Fr.Or Table)=}$$

$= -2 \cdot \frac{1}{2}\ln\left|\frac{u+1}{u-1}\right| + c = \ln\left|\frac{\sqrt{e^x+1}-1}{\sqrt{e^x+1}+1}\right| + c$ we can simplify

$$= c + \ln\left|\frac{\sqrt{e^x+1}-1}{\sqrt{e^x+1}+1} \cdot \frac{\sqrt{e^x+1}-1}{\sqrt{e^x+1}-1}\right| = c + \ln\frac{\left(\sqrt{e^x+1}-1\right)^2}{e^x} = c + 2\ln\left(\sqrt{e^x+1}-1\right) - x.$$