Exercises 7.3 Problem 5

Find the length of the curve $y^3 = x^2$ from $(-1,1)$ to $(1,1)$.

Solution:

$$y = x^{2/3}, \quad y' = f'(x) = \frac{2}{3}x^{-1/3}$$

Now sub this into the arc length integral formula.

$$s = \int_1^{-1} \sqrt{1 + \frac{4}{9}x^{-2/3}}\ dx.$$

Since $y = x^{2/3}$ is an odd function, ie, $f(-x) = -f(x)$, then the length of this curve on $x\epsilon(-1,0)$ is the same as the length of this curve on $x\epsilon(0,1)$. Thus,

$$s = 2\int_0^{1} \sqrt{1 + \frac{4}{9}x^{-2/3}}\ dx.$$

Now we need a trick, multiply the top and bottom of the integrand as follows,

$$s = 2\int_0^1 \frac{3\left|x\right|^{1/3}}{3\left|x\right|^{1/3}}\sqrt{1 + \frac{4}{9}x^{-2/3}}\ dx = 2\int_0^1 \frac{1}{3\left|x\right|^{1/3}}\sqrt{9\left|x\right|^{2/3} + 4}\ dx.$$

We can drop the absolute values because we are integrating over $x\epsilon(0,1)$,

$$s = 2\int_0^1 \frac{1}{3x^{1/3}}\sqrt{9x^{2/3} + 4}\ dx.$$

Let $u = 9x^{2/3} + 4 \quad \Rightarrow \quad du = 6x^{-1/3}dx.$

And so finally,

$$s = \int_{x=0}^{x=1} \frac{1}{9}\sqrt{u}\ du = \int_4^{13} \frac{1}{9}\sqrt{u}\ du = \frac{2}{27}\left(13^{3/2} - 4^{3/2}\right) = \frac{2\left(13^{3/2}\right) - 16}{27}.$$

This was a bit of a tough one, don't worry about it too much. Just understand it! Be sure to ask me if something doesn't make sense.