

Below is a list of some problems and solution hints for the section we are studying on ordinary differential equations. Since the purpose of these notes is to assist you in learning the material, not every detail is given. If the hint(s) provided are not sufficient for you to understand the problem, *it is essential that you talk to another human being* such as your professor to make sure that you do indeed understand the problem.

1. A mass m is thrown upward with initial velocity v_0 . Air resistance is proportional to its instantaneous velocity, the constant of proportionality being k . Show that the maximum height attained is

$$\frac{mv_0}{k} - \frac{m^2g}{k^2} \ln \left(1 + \frac{kv_0}{mg} \right).$$

SOLUTION. Let x be the position upwards, and v the velocity. From Newton's law the mathematical formulation of the differential equation we are trying to solve is

$$m \frac{dv}{dt} = -mg - kv \quad v(0) = v_0$$

Separating the variables, integrating, and substituting in the initial condition gives

$$v(t) = \frac{1}{k}((mg + kv_0)e^{-kt/m} - mg)$$

The time of maximum height occurs when $v(t) = 0$, and this implies that

$$e^{-kt/m} = \frac{mg}{mg + kv_0}$$

or that

$$t = \frac{m}{k} \ln \frac{mg + kv_0}{mg}.$$

Since $x(t) = \int v(t)dt$, the rest is straightforward.

2. A tank has 40 gal of pure water. A brine solution with 1 lb salt/gal enters at 2 gal/min, and the well-stirred mixture leaves at the same rate. When will the water leaving have 1/2 lb/gal salt?

SOLUTION. Let A denote the amount of salt in the tank at time t . The mathematical formulation of the differential equation we are trying to solve is

$$\frac{dA}{dt} = 2 - \frac{A}{20} \quad A(0) = 0.$$

This equation separates and integrates to give

$$A(t) = 40(1 - e^{-t/20}).$$

The concentration is $1/2$ when $A(t) = 20$, or $t = 20 \ln 2$.

3. Find the general solution of

$$(D^2 + 2D + 1)y = 2 \cos 2x + 3x + 2 + 3e^x$$

SOLUTION. We have a double root, so we get $(c_1 + c_2x)e^{-x}$ for the solution of the homogeneous equation. After a slog, I get

$$\frac{8}{25} \sin 2x - \frac{6}{25} \cos 2x + \frac{3}{2}e^x + 3x - 8$$

for the particular solution.

4. Solve

$$y'' + y = 6 \cos^2 x, \quad y(0) = 0, \quad y(\pi/2) = 0.$$

SOLUTION. It is easiest to rewrite the ode using $\cos^2 x = (1 + \cos 2x)/2$, and then use the method of undetermined coefficients to get

$$y(x) = c_1 \cos x + c_2 \sin x + 3 - \cos 2x$$

as the general solution. Now it is easy to solve for c_1 and c_2 .

5. Solve

$$(D^3 + D)y = x + \sin x + \cos x$$

SOLUTION. Since we get a real root and a complex pair, for the homogeneous solution we get

$$c_1 e^{-x} + e^{x/2}(c_2 \sin(\sqrt{3}x/2) + c_3 \cos(\sqrt{3}x/2))$$

and $x + \cos x$ for the particular solution. Adding them up gives the desired answer.

6. Solve

$$(D^4 - 1)y = \cosh x$$

SOLUTION. Since factoring the auxiliary equation gives $\pm 1, \pm i$ as roots, we get

$$y_c = c_1 e^x + c_2 e^{-x} + c_3 \sin x + c_4 \cos x$$

as the complementary solution and undetermined coefficients will give

$$y_p = -\frac{3}{8} \cosh x + \frac{1}{4} \sinh x$$

as the particular solution. Of course, you can do the whole thing with the exponentials e^x and e^{-x} using the definition of $\cosh x$ if you prefer.

7. Solve

$$y'' + 4y = \csc 2x.$$

SOLUTION. Using the method of variation of parameters, we do the required integrals and find

$$c_1 \sin 2x + c_2 \cos 2x + \frac{1}{4} \sin 2x \ln(\sin 2x) - \frac{1}{2} x \cos 2x$$

8. Use variation of parameters to solve

$$y' + P(x)y = Q(x).$$

SOLUTION. The only thing to note here is that the homogeneous solution is found by separating and integrating to get

$$y_c = ce^{-\int P(x) dx}$$

where c is an integration constant. We then assume that the general solution is of the form $y = y_c v(x)$, and substituting this into the original ode gives

$$y_c v' = Q(x)$$

because $y'_c v + P(x)y_c v = (y'_c + P(x)y_c)v = 0$. At any rate, we now get the usual formula when we solve for $v(x)$ by separating and integrating.