Below is a list of some problems and solution hints for the section we are studying on ordinary differential equations. Since the purpose of these notes is to assist you in learning the material, not every detail is given. If the hint(s) provided are not sufficient for you to understand the problem, *it is essential that you talk to another human* being such as your professor to make sure that you do indeed understand the problem.

1. Show that
$$y = \frac{1}{x}\int_1^x \frac{e^t}{t}\,dt$$
is a solution of $x^2y' + xy = e^x$.

   SOLUTION. This follows immediately from noticing that the fundamental theorem of calculus gives
$$y' = -\frac{1}{x^2}\int_1^x \frac{e^t}{t}\,dt + \frac{e^x}{x^2}.$$

2. Solve (a) $dy/dx = x^2\sqrt{y}$, (b) $dy/dx = e^{2x-y}/e^{x+y}$.

   SOLUTION. (a) separates and integrates as
$$\int \frac{1}{\sqrt{y}}\,dy = \int x^2\,dx.$$
(b) is rewritten using properties of exponentials as
$$\frac{dy}{dx} = \frac{e^{2x}e^{-y}}{e^x e^y} = \frac{e^x}{e^{2y}}$$
It is now clear that the equation separates.

3. Solve $2y\cos x\,dx + 3\sin x\,dy = 0$.

   SOLUTION. The equation separates as
$$2\cot x\,dx = -\frac{3}{y}\,dy$$

4. Solve
$$\frac{dy}{dx} = \frac{(y-1)(x-2)(y+3)}{(x-1)(y-2)(x+3)}.$$

SOLUTION. Once again, the equation immediately separates as

$$\frac{(y-2)}{(y-1)(y+3)}dy = \frac{(x-2)}{(x-1)(x+3)}dx.$$

The integration is routinely done by partial fractions.

5. Solve $dy/dx = F(y/x)$ by changing the dependent variable from y to v by $y = vx$. Use this method to solve

$$y' = \frac{y}{x} + \sec^2 \frac{y}{x}.$$

SOLUTION. The product rule gives $y' = v'x + v$, and so the differential equation becomes, after the change of variable,

$$v'x + v = F(v).$$

For the equation at hand, the method gives

$$v'x + v = v + \sec^2 v.$$

Cancel v from both sides, separate and integrate to get

$$\frac{v}{2} + \frac{1}{4}\sin 2v = \ln x + c,$$

where c is an integration constant. Substituting out $v = y/x$ gives the solution.

6. Solve $y' = \sqrt{2x + 3y}$.

SOLUTION. The idea here is to change to a new dependent variable by $v = 2x + 3y$. Then the resulting equation separates to

$$\frac{1}{2 + 3\sqrt{v}}\,dv = 2\,dx$$

The v-integral may be done by using the substitution $z^2 = v$.

7. Solve

$$y' = \frac{1}{x - 3y}.$$

SOLUTION. If we reverse the roles of the variables, so that x is now the dependent variable, the resulting equation is

$$\frac{dx}{dy} = x - 3y,$$

which is a linear differential equation for x. An integrating factor for this is $e^{\int -1 dy}$.

8. The equation

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

is called Bernoulli's differential equation. It is linear if $n = 0, 1$. Show that it can be solved by the change of variable $v = y^{1-n}$. As a specific case, solve

$$y' - y = xy^2.$$

SOLUTION. The recommended change of variable yields

$$y' = \frac{yv'}{(1-n)v},$$

so substituting in the differential equation gives

$$\frac{yv'}{(1-n)v} + Py = Q\frac{y}{v}.$$

Divide through by y, and then multiply through by $(1-n)v$, and you get

$$v' + (1-n)Pv = (1-n)Q,$$

which is a first order linear differential equation for v. Of course, you do not have to do the algebra exactly this way to get the answer.

As an application to the given equation, we see that $n = 2$, $P(x) = -1$, $Q(x) = x$, so we get

$$v' + v = -x.$$

An integrating factor for this equation is $e^{\int 1dx} = e^x$. We then find y by using that $v = 1/y$.