MATHEMATICS 253 L07 MIDTERM SOLUTIONS

March 10, 2005

1. Consider the function
$$f(x) = \ln\left(\frac{\pi}{2} + \arcsin(x)\right).$$
(a) Describe the domain and range of f. (b) Show that f is one-to-one, and describe the domain and range of f^{-1}, the inverse of f. (c) Find a formula for the inverse of f.

Solution: To compute $\arcsin(x)$ we need $-1 \le x \le +1$. But the range of $\arcsin(x)$ is $[-\frac{\pi}{2}, +\frac{\pi}{2}]$, and so the range of $\frac{\pi}{2} + \arcsin(x)$ will be $[0, \pi]$. Now to compute $\ln(u)$ we need $u > 0$, so we have to eliminate 0 from the range of $\frac{\pi}{2} + \arcsin(x)$, which means we have to eliminate $-\frac{\pi}{2}$ from the range of $\arcsin(x)$ so the domain of $f(x)$ will be $(-1, +1]$, and the range will be the range of $\ln(u)$ applied to $0 < u \le \pi$, thus the range is $(-\infty, \ln(\pi)]$.

(b) We calculate the derivative of f:
$$f'(x) = \frac{1}{\left(\frac{\pi}{2} + \arcsin(x)\right)} \cdot \frac{1}{\sqrt{1-x^2}},$$
which never vanishes, so by Rolles theorem the function is one-to-one. The domain of f^{-1} will be the range of f, while the range of f^{-1} will be the domain of f.

(c) We write $y = \ln\left(\frac{\pi}{2} + \arcsin(x)\right)$ and solve for x as a function of y:
$$e^y = \frac{\pi}{2} + \arcsin(x) \Rightarrow \arcsin(x) = e^y - \frac{\pi}{2} \Rightarrow x = \sin\left(e^y - \frac{\pi}{2}\right).$$
Thus $f^{-1}(x) = \sin\left(e^x - \frac{\pi}{2}\right)$.

2. Evaluate $\int x^4 \arcsin(x)\, dx$. Show all work, and state your answer in terms of the original variable of integration.

Solution: Kill the ugly function by integration by parts:
$$U = \arcsin(x), \quad dV = x^4\, dx, \quad dU = \frac{1}{\sqrt{1-x^2}}\, dx, \quad V = \frac{1}{5}x^5.$$
Then the integral becomes
$$\frac{1}{5}x^5 \arcsin(x) - \frac{1}{5}\int \frac{x^5}{\sqrt{1-x^2}}\, dx.$$
Now we need to evaluate
$$\int \frac{x^5}{\sqrt{1-x^2}}\, dx \,\Big|_{\substack{x=\sin(u) \\ dx=\cos(u)\, du}} = \int \frac{\sin^5(u)}{\cos(u)} \cos(u)\, du = \int \sin^4(u) \sin(u)\, du =$$

$$= \int [\sin^2]^2 \sin(u)\, du \,\Big|_{\substack{w=\cos(u)\\ dw=-\sin(u)\,du}} = \int [1-w^2]^2\, dw = w - \frac{2}{3}w^3 + \frac{1}{5}w^5 + C =$$

$$= \cos(u) - \frac{2}{3}\cos^3(u) + \frac{1}{5}\cos^5(u) + C = \sqrt{1-x^2} - \frac{2}{3}[1-x^2]^{\frac{3}{2}} + \frac{1}{5}[1-x^2]^{\frac{5}{2}} + C.$$

This last was got by drawing the triangle based on $x = \sin(u)$.

3. Evaluate $\int \frac{x+2}{[x^2+2x+5]^{\frac{3}{2}}}\, dx$.

   Show all work, and state your answer in terms of the original variable of integration

   Solution: We split the integral into two integrals in order to get an integral of the form $\int \frac{f'(x)}{[f(x)]^{\frac{3}{2}}}\, dx$ plus an integral that we can do using trig substitution (direct trig substitution in the original integral after completing the square–as below in the second integral– is also possible):

$$= \frac{1}{2}\int \frac{2x+2}{[x^2+2x+5]^{\frac{3}{2}}}\, dx + \int \frac{1}{[(x+1)^2+4]^{\frac{3}{2}}}\, dx =$$

$$= -\left[x^2+2x+5\right]^{-\frac{1}{2}} + \int \frac{1}{[(x+1)^2+4]^{\frac{3}{2}}}\, dx \,\Big|_{\substack{x+1=2\tan(u)\\ dx=2\sec^2(u)\,du}}.$$

   The remaining integral becomes

$$\int \frac{1}{[4\tan^2(u)+4]^{\frac{3}{2}}} 2\sec^2(u)\, du = \int \frac{1}{4\sec(u)}\, du =$$

$$\frac{1}{4}\int \cos(u)\, du = \frac{1}{4}\sin(u) + C = \frac{x+1}{\sqrt{4+(x+1)^2}}.$$

4. Evaluate $\int_3^4 \frac{5x^2-7x+4}{(x-2)(x^2+1)}\, dx$.

   Solution: Partial fractions:

$$\frac{5x^2-7x+4}{(x-2)(x^2+1)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1} \;\Rightarrow\; 5x^2-7x+4 = A(x^2+1) + (Bx+C)(x-2).$$

   Setting $x = 2$ yields $A = 2$. Then we get (setting $A = 2$):

$$3x^2 - 7x + 2 = Bx^2 + (C-2B)x - 2C \;\Rightarrow C = -1,\; B = 3.$$

   Thus our integral becomes

$$\int \frac{2}{x-2}\, dx + \int \frac{3x-1}{x^2+1}\, dx = 2\ln|x-2| + \frac{3}{2}\int \frac{2x}{x^2+1}\, dx - \int \frac{1}{x^2+1}\, dx =$$

$$= 2\ln|x-2| + \frac{3}{2}\ln(x^2+1) - \arctan(x) + C.$$

5. Evaluate

$$\int \frac{1}{x+2\sqrt{x-1}}\,dx.$$

Show all work, and state your answer in terms of the original variable of integration

Solution: Use the natural variable $u^2 = x - 1$:

$$= \int \frac{1}{u^2+1+2u} 2u\,du = 2\int \frac{u}{(u+1)^2}\,du \,|_{\substack{w=u+1 \\ dw=du}} =$$

$$= 2\int \frac{w-1}{w^2}\,dw = 2\int \left[\frac{1}{w} - \frac{1}{w^2}\right] dw = 2\left[\ln(|w|) + \frac{1}{w}\right] + C =$$

$$= 2\left[\ln(1+\sqrt{x-1}) + \frac{1}{1+\sqrt{x-1}}\right] + C.$$

6. Consider the bounded region determined by the curves $y = \arctan(x)$, $y = \frac{\pi}{4}x$, $0 \leq x \leq 1$. Write down the integral for the area of this region as a definite integral (a) on the x-axis, (b) on the y-axis. DO NOT EVALUATE, but be sure to put in limits. (Note: On the given interval, the graph of the line lies below the graph of $\arctan(x)$.)

Solution:

(a) $$\int_0^1 [\arctan(x) - \frac{\pi}{4}x]\,dx.$$

(b) $$\int_0^{\frac{\pi}{4}} [\frac{4}{\pi}y - \tan(y)]\,dy.$$