## First-Order Homogeneous Equations

19.2,

(1)

A first-order DE of the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

is said to be **homogeneous**. This is a *different* use of the term homogeneous from that in the previous section, which applied only to linear equations. Here homogeneous refers to the fact that $y/x$, and therefore $g(x, y) = f(y/x)$ is *homogeneous of degree 0* in the sense described after Example 7 in Section 13.5. Such a homogeneous equation can be transformed into a separable equation (and therefore solved) by means of a change of dependent variable. If we set

$$v = \frac{y}{x}, \qquad \text{or equivalently} \qquad y = xv(x),$$

then we have

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

and the original differential equation transforms into

$$\frac{dv}{dx} = \frac{f(v) - v}{x},$$

which is separable.

**EXAMPLE 4** Solve the equation

$$\frac{dy}{dx} = \frac{x^2 + xy}{xy + y^2}.$$

***SOLUTION*** The equation is homogeneous. (Divide the numerator and denominator of the right-hand side by $x^2$ to see this.) If $y = vx$ the equation becomes

$$v + x\frac{dv}{dx} = \frac{1 + v}{v + v^2} = \frac{1}{v},$$

or

$$x\frac{dv}{dx} = \frac{1 - v^2}{v}.$$

Separating variables and integrating, we calculate

$$\int \frac{v\,dv}{1 - v^2} = \int \frac{dx}{x} \qquad \text{let } u = 1 - v^2$$

$$-\frac{1}{2}\int \frac{du}{u} = \int \frac{dx}{x}$$

$$-\ln|u| = 2\ln|x| + C_1 = \ln C_2x^2 \qquad (C_1 = \ln C_2)$$

$$\frac{1}{|u|} = C_2x^2$$

$$|1 - v^2| = \frac{C_3}{x^2} \qquad (C_3 = 1/C_2)$$

$$\left|1 - \frac{y^2}{x^2}\right| = \frac{C_3}{x^2}.$$

The solution is best expressed in the form $x^2 - y^2 = C_4$. However, near points where $y \neq 0$, the equation can be solved for $y$ as a function of $x$. ■

## EXERCISES 19.2

Solve the differential equations in Exercises 1–20

1. $\dfrac{dy}{dx} = \dfrac{y}{2x}$
2. $\dfrac{dy}{dx} = \dfrac{3y-1}{x}$
3. $\dfrac{dy}{dx} = \dfrac{x^2}{y^2}$
4. $\dfrac{dy}{dx} = x^2y^2$
5. $\dfrac{dy}{dx} = \dfrac{x^2}{y^3}$
6. $\dfrac{dy}{dx} = x^2y^3$
7. $\dfrac{dY}{dt} = tY$
8. $\dfrac{dx}{dt} = e^x \sin t$
9. $\dfrac{dy}{dx} = 1 - y^2$
10. $\dfrac{dy}{dx} = 1 + y^2$
11. $\dfrac{dy}{dt} = 2 + e^y$
12. $\dfrac{dy}{dx} = y^2(1-y)$
13. $\dfrac{dy}{dx} = \sin x \cos^2 y$
14. $x\dfrac{dy}{dx} = y \ln x$
15. $\dfrac{dy}{dx} = \dfrac{x+y}{x-y}$
16. $\dfrac{dy}{dx} = \dfrac{xy}{x^2+2y^2}$
17. $\dfrac{dy}{dx} = \dfrac{x^2+xy+y^2}{x^2}$
18. $\dfrac{dy}{dx} = \dfrac{x^3+3xy^2}{3x^2y+y^3}$
19. $x\dfrac{dy}{dx} = y + x\cos^2\left(\dfrac{y}{x}\right)$
20. $\dfrac{dy}{dx} = \dfrac{y}{x} - e^{-y/x}$



Answers:

**Section 19.2 (page 962)**

1. $y^2 = Cx$
3. $x^3 - y^3 = C$
5. $\dfrac{y^4}{4} = \dfrac{x^3}{3} + C$
7. $Y = Ce^{t^2/2}$
9. $y = \dfrac{Ce^{2x}-1}{Ce^{2x}+1}$
11. $y = -\ln\left(Ce^{-2t} - \frac{1}{2}\right)$
13. $y = \tan^{-1}(C - \cos x) + n\pi$
15. $2\tan^{-1}(y/x) = \ln(x^2+y^2) + C$
17. $\tan^{-1}(y/x) = \ln|x| + C$
19. $y = x\tan^{-1}(\ln|Cx|)$
21. $y^3 + 3y - 3x^2 = 24$

Answers:

**Section 19.8 (page 993)**

1. $y = -\dfrac{1}{2} + C_1e^x + C_2e^{-2x}$
3. $y = -\dfrac{1}{2}e^{-x} + C_1e^x + C_2e^{-2x}$
5. $y = -\frac{2}{125} - \frac{4x}{25} + \frac{x^2}{5} + C_1e^{-x}\cos(2x) + C_2e^{-x}\sin(2x)$
7. $y = -\dfrac{1}{5}xe^{-2x} + C_1e^{-2x} + C_2e^{3x}$

9. $y = \dfrac{1}{8}e^x(\sin x - \cos x) + e^{-x}(C_1\cos x + C_2\sin x)$
11. $y = 2x + x^2 - xe^{-x} + C_1 + C_2e^{-x}$
15. $y_p = \dfrac{x^2}{3}, \quad y = \dfrac{x^2}{3} + C_1x + \dfrac{C_2}{x}$
17. $y = \dfrac{1}{2}x\ln x + C_1x + \dfrac{C_2}{x}$
19. $y = -x^2 + C_1x + C_2xe^x$

3

now consider the problem of solving the nonhomogeneous second-order differ-ial equation

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x). \qquad (*)$$

assume that two independent solutions, $y_1(x)$ and $y_2(x)$, of the corresponding hogeneous equation

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

are known. The function $y_h(x) = C_1y_1(x) + C_2y_2(x)$, which is the general solution of the homogeneous equation, is called the **complementary function** for the nonhomogeneous equation. Theorem 2 of Section 19.1 suggests that the general solution of the nonhomogeneous equation is of the form

$$y = y_p(x) + y_h(x) = y_p(x) + C_1y_1(x) + C_2y_2(x),$$

where $y_p(x)$ is any **particular solution** of the nonhomogeneous equation. All we need to do is find *one solution* of the nonhomogeneous equation and we can write the general solution.

There are two common methods for finding a particular solution $y_p$ of the nonhomogeneous equation $(*)$:

1. The method of undetermined coefficients, and
2. The method of variation of parameters.

The first of these hardly warrants being called a *method*; it just involves making an educated guess about the form of the solution as a sum of terms with unknown coefficients and substituting this guess into the equation to determine the coefficients. This method works well for simple DEs, especially ones with constant coefficients. The nature of the *guess* depends on the nonhomogeneous term $f(x)$, but can also be affected by the solution of the corresponding homogeneous equation. A few examples will illustrate the ideas involved.

**EXAMPLE 1** Find the general solution of $y'' + y' - 2y = 4x$.

***SOLUTION*** Because the nonhomogeneous term $f(x) = 4x$ is a first-degree polynomial, we "guess" that a particular solution can be found which is also such a polynomial. Thus we try

$$y = Ax + B, \qquad y' = A, \qquad y'' = 0.$$

Substituting these expressions into the given DE we obtain

$$\begin{aligned} 0 + A - 2(Ax + B) &= 4x \qquad \text{or} \\ -(2A + 4)x + (A - 2B) &= 0. \end{aligned}$$

This latter equation will be satisfied for all $x$ provided $2A + 4 = 0$ and $A - 2B = 0$. Thus we require $A = -2$ and $B = -1$; a particular solution of the given DE is

$$y_p(x) = -2x - 1.$$

Since the corresponding homogeneous equation $y'' + y' - 2y = 0$ has auxiliary equation $r^2 + r - 2 = 0$ with roots $r = 1$ and $r = -2$, the given DE has the general solution

$$y = y_p(x) + C_1e^x + C_2e^{-2x} = -2x - 1 + C_1e^x + C_2e^{-2x}.$$ ■

**EXAMPLE 2** Find general solutions of the equations (where $'$ denotes $d/dt$)

(a) $y'' + 4y = \sin t$,

(b) $y'' + 4y = \sin(2t)$,

(c) $y'' + 4y = \sin t + \sin(2t)$.

(4)

(a) Let us look for a particular solution of the form

**SOLUTION**

$$y = A\sin t + B\cos t \qquad \text{so that}$$
$$y' = A\cos t - B\sin t$$
$$y'' = -A\sin t - B\cos t.$$

Substituting these expressions into the DE $y'' + 4y = \sin t$, we get

$$-A\sin t - B\cos t + 4A\sin t + 4B\cos t = \sin t,$$

which is satisfied for all $x$ if $3A = 1$ and $3B = 0$. Thus $A = 1/3$ and $B = 0$. Since the homogeneous equation $y'' + 4y = 0$ has general solution $y = C_1\cos(2t) + C_2\sin(2t)$, the given nonhomogeneous equation has the general solution

$$y = \frac{1}{3}\sin t + C_1\cos(2t) + C_2\sin(2t).$$

(b) Motivated by our success in part (a), we might be tempted to try for a particular solution of the form $y = A\sin(2t) + B\cos(2t)$, but that won't work, because this function is a solution of the homogeneous equation, so we would get $y'' + 4y = 0$ for any choice of $A$ and $B$. In this case it is useful to try

$$y = At\sin(2t) + Bt\cos(2t).$$

We have

$$\begin{aligned} y' &= A\sin(2t) + 2At\cos(2t) + B\cos(2t) - 2Bt\sin(2t) \\ &= (A - 2Bt)\sin(2t) + (B + 2At)\cos(2t) \\ y'' &= -2B\sin(2t) + 2(A - 2Bt)\cos(2t) + 2A\cos(2t) \\ &\quad - 2(B + 2At)\sin(2t) \\ &= -4(B + At)\sin(2t) + 4(A - Bt)\cos(2t). \end{aligned}$$

Substituting into $y'' + 4y = \sin(2t)$ leads to

$$-4(B + At)\sin(2t) + 4(A - Bt)\cos(2t) + 4At\sin(2t) + 4Bt\cos(2t) = \sin(2t).$$

Observe that the terms involving $t\sin(2t)$ and $t\cos(2t)$ cancel out and we are left with

$$-4B\sin(2t) + 4A\cos(2t) = \sin(2t).$$

which is satisfied for all $x$ if $A = 0$ and $B = -1/4$. Hence, the general solution for part (b) is

$$y = -\frac{1}{4}t\cos(2t) + C_1\cos(2t) + C_2\sin(2t).$$

(c) Since the homogeneous equation is the same for (a), (b), and (c), and the nonhomogeneous term in equation (c) is the sum of the nonhomogeneous terms in equations (a) and (b), the sum of particular solutions of (a) and (b) is a particular solution of (c). (This is because the equation is *linear.*) Thus the general solution of equation (c) is

$$y = \frac{1}{3}\sin t - \frac{1}{4}t\cos(2t) + C_1\cos(2t) + C_2\sin(2t).$$

■

Summary:

To find a particular solution $y_p(x)$ of the second-order linear, constant-coefficient, nonhomogeneous DE

$$a_2\frac{d^2y}{dx^2}+a_1\frac{dy}{dx}+a_0y=f(x)$$

use the following forms:

If $f(x)=P_n(x)$ try $y_p=x^m A_n(x)$.
If $f(x)=P_n(x)e^{rx}$ try $y_p=x^m A_n(x)e^{rx}$.
If $f(x)=P_n(x)e^{rx}\cos(kx)$ try $y_p=x^m e^{rx}[A_n(x)\cos(kx)+B_n(x)\sin(kx)]$.
If $f(x)=P_n(x)e^{rx}\sin(kx)$ try $y_p=x^m e^{rx}[A_n(x)\cos(kx)+B_n(x)\sin(kx)]$.

where $m$ is the smallest of the integers 0, 1, and 2, that ensures that no term of $y_p$ is a solution of the corresponding homogeneous equation

$$a_2\frac{d^2y}{dx^2}+a_1\frac{dy}{dx}+a_0y=0.$$

## Variation of Parameters

A more formal method for finding a particular solution $y_p(x)$ of the nonhomogeneous equation when we know two independent solutions, $y_1(x)$ and $y_2(x)$, of the homogeneous equation is to replace the constants in the complementary function by functions, that is, search for $y_p$ in the form

$$y_p=u_1(x)y_1(x)+u_2(x)y_2(x).$$

Requiring $y_p$ to satisfy the given nonhomogeneous DE provides one equation that must be satisfied by the two unknown functions $u_1$ and $u_2$. We are free to require them to satisfy a second equation also. To simplify the calculations below, we choose this second equation to be

$$u_1'(x)y_1(x)+u_2'(x)y_2(x)=0.$$

Now we have

$$\begin{aligned} y_p' &= u_1'y_1+u_1y_1'+u_2'y_2+u_2y_2'=u_1y_1'+u_2y_2' \\ y_p'' &= u_1'y_1'+u_1y_1''+u_2'y_2'+u_2y_2''. \end{aligned}$$

Substituting these expressions into the given DE we obtain

$$\begin{aligned} a_2(u_1'y_1'+u_2'y_2') &+ u_1(a_2y_1''+a_1y_1'+a_0y_1)+u_2(a_2y_2''+a_1y_2'+a_0y_2) \\ &= a_2(u_1'y_1'+u_2'y_2')=f(x), \end{aligned}$$

because $y_1$ and $y_2$ satisfy the homogeneous equation. Therefore $u_1'$ and $u_2'$ satisfy the pair of equations

$$\begin{aligned} u_1'(x)y_1(x)+u_2'(x)y_2(x) &= 0 \\ u_1'(x)y_1'(x)+u_2'(x)y_2'(x) &= \frac{f(x)}{a_2(x)}. \end{aligned}$$

We can solve these two equations for the unknown functions $u_1'$ and $u_2'$ by Cramer's Rule (Theorem 5 of Section 12.5), or otherwise, and obtain

$$u_1'=-\frac{y_2(x)}{W(x)}\frac{f(x)}{a_2(x)}, \qquad u_2'=\frac{y_1(x)}{W(x)}\frac{f(x)}{a_2(x)},$$

where $W(x)$, called the **Wronskian** of $y_1$ and $y_2$, is the determinant

$$W(x)=\begin{vmatrix} y_1(x) & y_2(x) \end{vmatrix}$$

**EXAMPLE 3** Find the general solution of $y'' - 3y' + 2y = 4x$.

***SOLUTION*** First we solve the homogeneous equation $y'' - 3y' + 2y = 0$, which has auxiliary equation $r^2 - 3r + 2 = 0$ with roots $r = 1$ and $r = 2$. Therefore two independent solutions of the homogeneous equation are $y_1 = e^x$ and $y_2 = e^{2x}$, and the complementary function is

$$y_h = C_1 e^x + C_2 e^{2x}.$$

A particular solution $y_p(x)$ of the nonhomogeneous equation can be found in the form

$$y_p = u_1(x)e^x + u_2(x)e^{2x}$$

where $u_1$ and $u_2$ satisfy

$$\begin{aligned} u_1' e^x + 2u_2' e^{2x} &= 4x \\ u_1' e^x + u_2' e^{2x} &= 0. \end{aligned}$$

We solve these linear equations for $u_1'$ and $u_2'$ and then integrate to obtain

$$\begin{aligned} u_1' &= -4xe^{-x} & u_2' &= 4xe^{-2x} \\ u_1 &= 4(x + 1)e^{-x} & u_2 &= -(2x + 1)e^{-2x}. \end{aligned}$$

Hence $y_p = 4x + 4 - (2x + 1) = 2x + 3$ is a particular solution of the nonhomogeneous equation, and the general solution is

$$y = 2x + 3 + C_1 e^x + C_2 e^{2x}.$$

***REMARK*** This method for solving the non-homogeneous equation is called the **method of variation of parameters.** It is completely general and extends to higher order equations in a reasonable way, but it is computationally somewhat difficult. We could have found $y_p$ more easily had we "guessed" that it would be of the form $y_p = Ax + B$ and substituted this into the differential equation to get

$$\begin{aligned} & -3A + 2(Ax + B) = 4x \\ \text{or} \quad & 2Ax + (2B - 3A) = 4x. \end{aligned}$$

The only way this latter equation can be satisfied for all $x$ is to have $2A = 4$ and $2B - 3A = 0$, that is, $A = 2$ and $B = 3$.

## EXERCISES 19.8

Find general solutions for the nonhomogeneous equations in Exercises 1–12 by the method of undetermined coefficients.

1. $y'' + y' - 2y = 1$
2. $y'' + y' - 2y = x$
3. $y'' + y' - 2y = e^{-x}$
4. $y'' + y' - 2y = e^x$
5. $y'' + 2y' + 5y = x^2$
6. $y'' + 4y = x^2$
7. $y'' - y' - 6y$ [illegible]
8. $y'' + 4y' + 4y = e^{-2x}$
9. $y'' + 2y' + 2y = e^x \sin x$
10. $y'' + 2y' + 2y = e^{-x} \sin x$
11. $y'' + y' = 4 + 2x + e^{-x}$
12. $y'' + 2y' + y = xe^{-x}$
13. Repeat Exercise 3 using the method of variation of parameters.
14. Repeat Exercise 4 using the method of variation of parameters.
15. Find a particular solution of the form $y = Ax^2$ for the Euler equation $x^2y'' + xy' - y = x^2$, and hence obtain the general solution of this equation on the interval $(0, \infty)$.
16. For what values of $r$ can the Euler equation $x^2y'' + xy' - y = x^r$ be solved by the method of the previous exercise. Find a particular solution for each such $r$.