FINAL Handout
MATH 253

1. Derive the formula for the volume of

   (a) a shere with radius R $\quad\left(V = \frac{4}{3}\pi R^3\right)$

   (b) a cone with radius R and the height H $\quad\left(V = \frac{1}{3}\pi R^2 H\right)$

   Solution for a)

   we can get a sphere by rotating the region = half of a circlular disk

   $x^2 + y^2 \le R^2, y > 0$ around the x-axis

   so $y = \sqrt{R^2 - x^2}, x \in [-R, R]$ and using slices

   $V = \pi \int\limits_{-R}^{R} \left(\sqrt{R^2 - x^2}\right)^2 dx = \pi \int\limits_{-R}^{R} (R^2 - x^2)\, dx = 2\pi \int\limits_{0}^{R} (R^2 - x^2)\, dx = 2\pi \left[R^3 - \frac{R^3}{3}\right] =$
   $\frac{4}{3}\pi R^3.$

   for b)

   we can get a cone by rotating a triangle around x- axis slices

   the region is below the line $y = \dfrac{R}{H}x$ for $x \in [0, H]$

   so $V = \pi \int\limits_{0}^{H} \left(\dfrac{R}{H}x\right)^2 dx = \pi \frac{R^2}{H^2} \left[\frac{x^3}{3}\right]_0^H = \frac{\pi}{3} R^2 H$

   OR around $y-$ *axis* but the region is *below* the line $y = \frac{H}{R}(R - x)$, $x \in [0, R]$

   so shells

   $V = 2\pi \int\limits_{0}^{R} x \frac{H}{R}(R - x)\, dx = 2\pi \frac{H}{R} \int\limits_{0}^{R} (Rx - x^2)\, dx = 2\pi \left[H\frac{x^2}{2} - \frac{H}{R}\frac{x^3}{3}\right]_0^R =$

   $= \pi H \left(R^2 - \frac{2}{3}R^2\right) = \frac{\pi}{3} R^2 H.$

2. Derive the formula for circumference of a circle with radius R. $(c = 2\pi R)$

   Sol.

   we can calculate the arclength of the top half of a circle first and then $c = 2c^+$

   for $y = \sqrt{R^2 - x^2}, x \in [-R, R]$ we can use the formula $c^+ = \int\limits_{-R}^{R} \sqrt{1 + (y')^2} dx$

   where $y' = \dfrac{-2x}{2\sqrt{R^2 - x^2}}$,$1 + (y\prime)^2 = \dfrac{R^2}{R^2 - x^2}$ thus

   $c = 2c^+ = 2R \int\limits_{-R}^{R} \dfrac{dx}{\sqrt{R^2 - x^2}} = 2R \left[\arcsin \frac{x}{R}\right]_{-R}^{R} = 2R \left[\arcsin 1 - \arcsin(-1)\right] =$

   $= 2r \left[\frac{\pi}{2} + \frac{\pi}{2}\right] = 2R\pi$

3. Find the arclength of the curve $(y-1)^2 = (1-x)^3$ between $P(0,2)$ and $R(1,1)$.

Sol.

first an explicit fomula for the curve ......$y = 1 + (1-x)^{\frac{3}{2}}$ for $x \in [0,1]$

($y-1$ is positive,since $y \geq 1$)

now, derivative ......$y' = \frac{3}{2}(1-x)^{\frac{1}{2}} \cdot (-1)$ and ...$1 + (y')^2 = 1 + \frac{9}{4}(1-x) = \frac{13-9x}{4}$

and the length

$$l = \int_0^1 \sqrt{1+(y')^2}dx = \int_0^1 \sqrt{\tfrac{13-9x}{4}}dx = \text{(by subst. } u = 13-9x, du = -9dx, u = 13, 4)$$

$$= \frac{1}{9}\cdot\frac{1}{2}\int_4^{13}\sqrt{u}du = \frac{1}{27}\left[u^{\frac{3}{2}}\right]_4^{13} = \frac{1}{27}\left[13\sqrt{13}-8\right].$$

4. Find the domain of definition of $f(x) = \sqrt{9-x^2}$ and then find the antiderivative $F(x) = \int f(x)dx$ - NOT using Tables.(Area of a circle)

Sol.

for the domain solve $\quad 9 - x^2 \geq 0 \quad 9 \geq x^2 \quad 3 \geq |x|$ so $x \in [-3,3] = D_f$

OR

$(3-x)(3+x) \geq 0$ testing$--^{neg}-_{-3}--^{pos}---_{3}--^{neg}--$

to find $F(x)$ we can use by parts or subst.

$$F(x) = \int 1\cdot\sqrt{9-x^2}dx = x\sqrt{9-x^2} - \int x\cdot\tfrac{-2x}{2\sqrt{9-x^2}}dx = x\sqrt{9-x^2} - \int \frac{9-x^2-9}{\sqrt{9-x^2}}dx =$$

(split the integrand)=

$$= x\sqrt{9-x^2} - \int \tfrac{9-x^2}{\sqrt{9-x^2}}dx + 9\int \tfrac{1}{\sqrt{9-x^2}}dx = x\sqrt{9-x^2} - F(x) + 9\arcsin\tfrac{x}{3} + c$$

(derivative of arcsin)

so now we'v got an equation for unknown function $F(x)$:

$2F(x) = x\sqrt{9-x^2} + 9\arcsin\frac{x}{3} + c$ thus $\quad F(x) = \frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\arcsin\frac{x}{3} + C$

OR by inverse subst.

$x = 3\sin t \quad dx = 3\cos t \quad$ for $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\quad t = \arcsin\frac{x}{3}$ so

$\int \sqrt{9-x^2}dx = 3\int\sqrt{9\left(1-\sin^2 t\right)}\cos tdt = 9\int\cos^2 tdt =$(double angle formula) $= 9\int\frac{1+\cos 2t}{2}dt =$

$= \frac{9}{2}\int dt + \frac{9}{2}\int\cos 2tdt = \frac{9}{2}t + \frac{9}{2}\cdot\frac{\sin 2t}{2} + c = \frac{9}{2}t + \frac{9}{2}\sin t\cos t + c$(back to x)

$= \frac{9}{2}\arcsin\frac{x}{3} + \frac{9}{2}\cdot\frac{x}{3}\frac{\sqrt{9-x^2}}{3} + c$ as above since $3\cos t = \sqrt{9-x^2}$

5. Approximate $\arcsin\frac{1}{3}$ using the Taylor polynomial of third degree T_3 centered at 0.

$f(x) = \arcsin x$ so $\quad a_0 = f(0) = 0$

$f'(x) = \frac{1}{\sqrt{1-x^2}}$ so $\quad a_1 = f'(0) = 1$

$f''(x) = -\frac{1}{2}\left(1-x^2\right)^{-\frac{3}{2}}(-2x) = x\left(1-x^2\right)^{-\frac{3}{2}}$ so $a_2 = 0$

$f'''(x) =$ (P.R.)$=\left(1-x^2\right)^{-\frac{3}{2}} + x\left(-\frac{3}{2}\right)\left(1-x^2\right)^{-\frac{5}{2}}(-2x)$

so $\quad a_3 = \frac{f'''(0)}{3!} = \frac{1}{6}$ and $T_3(x) = x + \frac{1}{6}x^3$

thus $\arcsin\frac{1}{3} \doteq T_3(\frac{1}{3}) = \frac{1}{3} + \frac{1}{6\cdot 27} = \dfrac{55}{162}$.

6. Find the general solution of $x^2y' - 4y = x^3 \cdot \ln x \cdot e^{-4/x}$.

Sol.

first order linear but first rewrite:

for $x > 0 \quad y' - \frac{4}{x^2}y = x\ln x \cdot e^{-\frac{4}{x}}$ so $p(x) = -\frac{4}{x^2}$ and $P(x) = \int p(x)dx = \frac{4}{x}$

and $\mu = e^{P(x)} = e^{\frac{4}{x}}$

multiply by the integrating factor the rewritten equation

$e^{\frac{4}{x}} \cdot y' - \frac{4}{x^2}e^{\frac{4}{x}}y = x\ln x$ check whether the left-hand side $=(ye^{\frac{4}{x}})' = x\ln x$

integrate $\quad ye^{\frac{4}{x}} = \int x\ln x dx + c$

(by parts)$\int x\ln x dx = \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\cdot\frac{1}{x}dx = \frac{x^2}{2}\ln x - \frac{1}{2}\int x dx =$

$= \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + c \qquad ye^{\frac{4}{x}} = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + c$

and finally solve for $y \qquad y = e^{-\frac{4}{x}}\left(\frac{1}{2}x^2\ln x - \frac{1}{4}x^2\right) + ce^{-\frac{4}{x}}$ for $x \in (0, +\infty)$.

7. Solve the initial value problem

$$y'' + 4y' + 4y = 8x^2, \qquad y(0) = 2, \quad y'(0) = 4..$$

Sol.

second order ,linear,non-homog.,constant coeff.

first the corresponding homog. equation

$y'' + 4y' + 4y = 0$.......$r^2 + 4r + 4 = (r+2)^2 = 0$...$r_1 = r_2 = -2, , , y_1 = e^{-2x}, y_2 = xe^{-2x}$

and general sol for homog. $\qquad y_h = C_1e^{-2x} + C_2xe^{-2x}$

particular solution corresp.to the given $f(x) = 8x^2$could be found in the form

$y_p = ax^2 + bx + c$

$y_p' = 2ax + b$ and $y_p'' = 2a$.......back to the equation

$L(y_p) = 2a + 8ax + 4b + 4ax^2 + 4bx + 4c = 4ax^2 + (8a+4b)x + 4c + 2a + 4b = 8x^2 + 0x + 0$

compare coefficients of $x^2: \quad 4a = 8 \qquad a = 2$

of x^1 : $(8a+4b)=0 \quad b=-2a \qquad b=-4$

and finally of x^0 : $4c+2a+4b=0 \quad c=-b-\frac{a}{2} \quad c=3$..back to $\quad y_p=2x^2-4x+3$

Together $\qquad y=C_1e^{-2x}+C_2xe^{-2x}+2x^2-4x+3$ general solution,

finally the conditions $x=0, y=2 \qquad 2=C_1+3 \rightarrow C_1=-1$

differentiate $\quad y'=-2C_1e^{-2x}+C_2(e^{-2x}-2xe^{-2x})+4x-4$

$x=0, y'=4 \quad 4=2+C_2-4 \rightarrow C_2=6$

Therefore the solution of the intial-value problem is

$y=-e^{-2x}+6xe^{-2x}+2x^2-4x+3.$

8. Find the general solution of the differential equation

$$y''+9y=10\sin 2x+e^{-x}$$

Sol.

second order,linear,non-homog

first,homog

$y''+9y=0$ constant coeff.,so solve $r^2+9=0, r_{1,2}=\pm 3i \quad y_1=\sin 3x, y_2=\cos 3x$

general sol of homog.equation is $\qquad y_h=c_1\sin 3x+c_2\cos 3x$

now

$f(x)=f_1(x)+f_2(x)$ where $f_1(x)=10\sin 2x$ and $f_2(x)=e^{-x}$

so particular solutions could be found in the form

$y_{p_1}=a\sin 2x+b\cos 2x$ and $y_{p_2}=Ae^{-x}$

now $\quad y'_{p_2}=-Ae^{-x}$ and $\quad y''_{p_2}=Ae^{-x}$

$L(y_{p_2})=10Ae^{-x}=f_2(x)=e^{-x}$ so $A=\frac{1}{10}$ and $y_{p_2}=\frac{1}{10}e^{-x}$;

similarly,

$y'_{p_1}=2a\cos 2x-2b\sin 2x$ and $y''_{p_1}=-4a\sin 2x-2b\cos 2x$

$L(y_{p_1})=-4a\sin 2x-2b\cos 2x+9a\sin 2x+9b\cos 2x=f_2(x)=10\sin 2x$

compare the coefficients of sin and cos:

$5a\sin 2x+5b\cos 2x=10\sin 2x+0\cos 2x$

$5a=10$ and $5b=0$ so $a=2$ and $b=0$ and $y_{p_1}=2\sin 2x.$

Together

$y=y_h+y_{p_1}+y_{p_2}=c_1\sin 3x+c_2\cos 3x+2\sin 2x+\frac{1}{10}e^{-x}.$

9. Find the domain and antiderivative of the following functions:

a. $\quad x\arcsin(2x)$

for domain solve $-1\le 2x\le 1$ so $x\in\left[-\frac{1}{2},\frac{1}{2}\right]$ and use subst. $2x=u$ and by parts for

$$\int x\arcsin(2x)\,dx = \tfrac{1}{4}\int u\arcsin u\ du = \tfrac{1}{4}\left(\frac{u^2}{2}\arcsin u\right) - \tfrac{1}{8}\int u^2\cdot\frac{1}{\sqrt{1-u^2}}du$$

(Table)=

$= \frac{1}{8}u^2\arcsin u + \frac{u}{16}\sqrt{1-u^2} - \frac{1}{16}\arcsin u + c = \frac{1}{2}x^2\arcsin 2x + \frac{x}{8}\sqrt{1-4x^2} - \frac{1}{16}\arcsin 2x + \mathrm{c}$

b. $\dfrac{x^2+2}{x-x^2} = -\dfrac{x^2+2}{x^2-x} = -\dfrac{x^2-x+x+2}{x^2-x} = -1-\dfrac{x+2}{x^2-x} = -1-\dfrac{x+2}{x(x-1)}$

so domain is any $x \neq 0,1$ by Partial Fractions

$\dfrac{x+2}{x(x-1)} = \dfrac{A}{x} + \dfrac{B}{(x-1)}$ $\quad (*) \quad$ $x+2 = A(x-1)+Bx$ so $x=0$ gives $A=-2$

and $x=1$ gives $B=3$ then

$$\int \frac{x^2+2}{x-x^2}dx = \int\left[-1-\frac{x+2}{x(x-1)}\right]dx = -x+2\ln|x|-3\ln|x-1|+c$$

c. $x\ln(2x+3)$

for domain $\quad 2x+3>0 \quad\quad x>-\frac{3}{2}$

use subst. $u=2x+3, du=2dx, x=\frac{1}{2}(u-3)$ then by parts

$$\int x\ln(2x+3)\,dx = \tfrac{1}{4}\int(u-3)\ln u\ du =$$

$$= \tfrac{1}{4}\left(\tfrac{1}{2}u^2-3u\right)\ln u - \tfrac{1}{4}\int\left(\tfrac{1}{2}u^2-3u\right)\tfrac{1}{u}du =$$

$$= \left(\tfrac{1}{8}u^2-\tfrac{3}{4}u\right)\ln u - \tfrac{1}{16}u^2+\tfrac{3}{4}u+c = (\text{ back to } x)$$

$$= \left[\tfrac{1}{8}(2x+3)^2-\tfrac{3}{4}(2x+3)\right]\ln(2x+3) - \tfrac{1}{16}(2x+3)^2+\tfrac{3}{4}(2x+3)+c$$

d. $\dfrac{1}{\sqrt{e^x+1}}$ $\quad\quad e^x>0$ always so domain any x

use inverse subst. $u=\sqrt{e^x+1}, du=\frac{1}{2\sqrt{e^x+1}}e^x dx$ OR

$u^2-1=e^x, \ln(u^2-1)=x, dx=2u\frac{du}{u^2-1}$ so

$$\int\frac{1}{\sqrt{e^x+1}}dx = \int\frac{1}{e^x\sqrt{e^x+1}}e^x dx = \int\frac{2}{u^2-1}dx \text{ (Part.fr.Or table)=}$$

$= -2\cdot\frac{1}{2}\ln\left|\frac{u+1}{u-1}\right|+c = \ln\left|\frac{\sqrt{e^x+1}-1}{\sqrt{e^x+1}+1}\right|+c$ we can simplify

$$= c+\ln\left|\frac{\sqrt{e^x+1}-1}{\sqrt{e^x+1}+1}\cdot\frac{\sqrt{e^x+1}-1}{\sqrt{e^x+1}-1}\right| = c+2\ln\left(\sqrt{e^x+1}-1\right)-x.$$