Final Exam Review Questions, Math 253 L07/L08

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1. Domain and range of a function, proving a function is one-to-one, domain and range and derivative of inverse function, formula for inverse function.

   Typical Problems:

   (a) If $f(x) = e^{\sqrt{4-x}}$, find the domain and range of f. Show f is one-to-one on its domain. Find a formula for the inverse function f^{-1}.(Handout 1)

   (b) Same as previous problem, for $f(x) = \ln\left[\frac{1}{1-x}\right]$. (Handout 1)

   (c) Same as above for $f(x) = \arcsin(\sqrt{3+x} - 2)$ (Ans.: Domain is $[-2,\ 6]$, range is $-1 \le x \le +1$.

2. Integration by parts. Three topics: Kill a power of x, kill the ugly function, find a recursion formula (reduction formula) for a set of integrals.

   (a) Evaluate $\int x\,(\ln(x))^2\,dx$. (Handout 3)

   (b) Evaluate without Table: $\int_0^{\frac{1}{2}} \arctan(2x)\,dx$. (Handout 3)

   (c) Evaluate $\int_{-1}^{0} x^2 e^{3x}\,dx$ (Handout 3).

   (d) Given $\int x^n e^{x^2}\,dx$, $n = 0, 1, 2, \ldots$, call these integrals I_n, e.g., $I_3 = \int x^3 e^{x^2}\,dx$. Show that
   $$I_n = \frac{1}{2}x^{n-1}e^{x^2} - \frac{n-1}{2}\int I_{n-2}.$$
   Given that $I_1 = \frac{1}{2}e^{x^2} + C$, use this recursion formula to find I_3 and I_5.
   (Ans.: To prove the formula, use parts with $U(x) = x^{n-1}$ and $dV(x) = xe^{x^2}\,dx$.
   $$I_3 = \frac{1}{2}x^2e^{x^2} - \frac{1}{2}e^{x^2}, \qquad I_5 = \frac{1}{2}x^4e^{x^2} - 2I_3.$$

   (e) Evaluate without table $\int e^{3x}\sin\left(\frac{x}{3}\right)\,dx$. (Handout 3)

3. Substitution, trig powers, trig substitution, inverse trig functions, completing the square, recoginizing logarithms ($\int \frac{f'(x)}{f(x)}\,dx$) , natural variable.
   Evaluate (Handout 3):

   (a)
   $$\int \frac{1}{2 + x^{\frac{1}{3}}}\,dx.$$

   (b)
   $$\int_0^{\frac{1}{2}} \frac{2x+1}{4x^2+1}\,dx.$$

(c)

$$\int_0^2 x\sqrt{4x - x^2}\,dx.$$

The following are from various sources:

(d)

$$\int \frac{\ln(x)}{x[1+\ln(x)]}\,dx.$$

(e)

$$\int \tan^2 \sec^2\,dx.$$

(f)

$$\int x^2 \arcsin(x)\,dx.$$

(g)

$$\int \frac{e^x}{1+e^{2x}}\,dx.$$

(h)

$$\int \frac{e^{2x}}{1+e^x}\,dx.$$

(i)

$$\int \cos^2(x)\sin^{101}(x)\,dx.$$

(j)

$$\int \frac{1}{[x^2+6x+11]^{\frac{3}{2}}}\,dx.$$

(k)

$$\int \sqrt{1+x-x^2}\,dx.$$

See also old quizzes and worksheets and the drill problems in the detailed syllabus.

4. Partial Fractions

Evaluate:

(a)

$$\int \frac{x^3+x^2-2}{(x-1)^2)\,(x^2+1)}\,dx.$$

(b)

$$\int \frac{3x-1}{x(x-1)(x+1)}\,dx.$$

5. Improper Integrals

Explain why the integral is improper, and decide whether convergent or divergent:

(a)
$$\int_0^{\frac{\pi}{2}} \sec^2(x)\,dx.$$

(b)
$$\int_0^{\infty} \sin(x)\,dx.$$

(c)
$$\int_{-1}^{1} \frac{x}{\sqrt{x+1}}\,dx.$$

(d)
$$\int_1^{\infty} \frac{1}{x^2+1}\,dx.$$

6. Volume by discs and shells. Consider the region bounded by $y = x + \ln(x)$ and the x-axis, $1 \leq x \leq e$. Find the volume of the solid obtained by

(a) rotating this region about the x-axis, using discs;

(b) rotating this region about the y-axis, using shells;

(c) rotating this region about the line $y = 9$, using discs;

(d) rotating this region about the line $x = 3$, using shells.

7. Arclength and surface area.

(a) Write down the integral for the arclength of the curve defined by $x^2y + x = 1$.

(b) Write down the integral for the surface area obtained by rotating the curve from the previous problem about the y-axis.

(c) Same as above, but rotate about the line x = -2.

(d) Same as above, but rotoate about the x-axis.

8. Taylor Polynomials (see Handout 5)

(a) Find the Taylor Polynomial $P_3(x)$ about $a = \frac{\pi}{2}$ for $f(x) = \ln(\sin(x))$.

(b) Derive the Taylor Polynomial of any degree for $f(x) = \sin(x)$ about $x = 0$.

9. Ordinary Differential Equations
Solve:

(a) $y'(x) + y(x)\sin x = \cos(x)\sin(x), \qquad y(2) = 3.$ (Handout 5)

(b) Find the general solution, in explicit form, of $(x+1)y' + (1+y)x^2 = 0$. (Handout 5)

(c) Find the solution, in the simplest form, of the initial value problem $y' = \frac{y}{x-y}, \quad y(2) = 1$. (Handout 5)

(d) Find the general solution of $y'(x) - 2xy(x) = x$. (Handout 5)

(e) $(x^2 + 1)\, y' + y = 0$.

(f) $y' - 3y = e^{2x}, \qquad y(0) = 4$.

(g) $y''(x) - 5y'(x) + 6y(x) = 0$.

(h) $y''(x) - 5y'(x) + 6y(x) = 3e^{4x}$.

(i) $y''(x) - 5y'(x) + 6y(x) = 2x^2 + 3x - 5$.

(j) $y''(x) - 4y'(x) + 4y(x) = 0$.

(k) $2y''(x) - 3y'(x) + 2y(x) = 0$

(l) $y''(x) - y(x) = \cos(3x)$.