

MATH 253
Handout # 3

1. SOLUTION

For 1)

$$\int_1^e x(\ln x)^2 dx = (\text{by parts: } du = x dx \dots \text{integrate } \dots u = \frac{x^2}{2},$$

$$v = \ln^2 x \dots \text{differentiate } \dots dv = 2 \ln x \cdot \frac{1}{x} dx)$$

$$= \left[\frac{x^2}{2} (\ln x)^2 \right]_1^e - \int_1^e \frac{x^2}{2} 2(\ln x) \frac{1}{x} dx = \frac{e^2}{2} \cdot 1 - 0 - \int_1^e x(\ln x) dx =$$

$$(\text{ by parts again: } du = x dx \dots u = \frac{x^2}{2}, v = \ln x \dots dv = \frac{1}{x} dx)$$

$$= \frac{e^2}{2} - \left[\frac{x^2}{2} \ln x \right]_1^e + \int_1^e \frac{x^2}{2} \cdot \frac{1}{x} dx = \frac{e^2}{2} - \frac{e^2}{2} + 0 + \frac{1}{2} \left[\frac{x^2}{2} \right]_1^e = \frac{1}{4} [e^2 - 1].$$

For 2)

$$\text{Use inverse subst. } u = 2 + \sqrt[3]{x} \quad (u - 2)^3 = x \quad 3(u - 2)^2 du = dx$$

$$\int \frac{1}{2 + \sqrt[3]{x}} dx = \int \frac{3(u - 2)^2}{u} du = 3 \int \frac{u^2 - 4u + 4}{u} du = 3 \int u du - 12 \int \frac{1}{u} du + 12 \int \frac{1}{u} du =$$

$$= \frac{3}{2} u^2 - 12u + 12 \ln |u| + c \quad (\text{for any } u \neq 0)$$

$$= \frac{3}{2} (2 + \sqrt[3]{x})^2 - 12(2 + \sqrt[3]{x}) + 12 \ln |2 + \sqrt[3]{x}| + c$$

and for the domain

$$u \neq 0 \text{ implies that } x \neq -8 \text{ since } (u - 2)^3 = x.$$

For 3)

$$\text{Complete the square: } -(x^2 - 6x - 7) = -[(x - 3)^2 - 16],$$

so it is a parabola open down

$$\text{with two roots } x = -1, 7 \text{ thus domain is } (-1, 7)$$

by substitution $u = x - 3, dx = du, x = u + 3$, we get

$$\int \frac{x}{\sqrt{7 + 6x - x^2}} dx = \int \frac{3 + u}{\sqrt{16 - u^2}} du = 3 \int \frac{1}{\sqrt{16 - u^2}} du + \int \frac{u}{\sqrt{16 - u^2}} du =$$

$$= 3 \arcsin \frac{u}{4} - \frac{1}{2} \int \frac{dv}{\sqrt{v}} \quad (\text{using } v = 16 - u^2, dv = -2u du \quad u \in (-4, 4)) =$$

$$= 3 \arcsin \frac{u}{4} - \sqrt{v} + c = 3 \arcsin \frac{x-3}{4} - \sqrt{16 - u^2} + c =$$

$$= 3 \arcsin \frac{x-3}{4} - \sqrt{7+6x-x^2} + c \quad \text{for } x \in (-1, 7)$$

For 4)

by parts:

$$du = 1dx. \text{ by integrating } u = x, v = \arctan 2x. \text{ by differentiating } dv = \frac{2dx}{1+(2x)^2}$$

$$F(x) = \int \arctan(2x) \cdot 1dx = x \cdot \arctan(2x) - \int x \cdot \frac{2}{1+4x^2} dx =$$

$$= x \arctan 2x - \int \frac{2xdx}{1+4x^2} = x \arctan 2x - \frac{1}{4} \int \frac{1}{u} du$$

$$(\text{ by substitution } u = 1 + 4x^2, du = 8xdx) = x \arctan 2x - \frac{1}{4} \ln(1 + 4x^2)$$

$$\text{so integral } I = F\left(\frac{1}{2}\right) - F(0) = \frac{1}{2} \arctan 1 - \frac{1}{4} \ln 2 = \frac{\pi}{8} - \frac{1}{4} \ln 2$$

(since $\arctan 0 = 0, \ln 1 = 0, c = 0$).

For 5)

$$I = \int_0^{\frac{1}{2}} \frac{2x+1}{4x^2+1} dx = \int_0^{\frac{1}{2}} \frac{2x}{4x^2+1} dx + \int_0^{\frac{1}{2}} \frac{1}{4x^2+1} dx = \frac{1}{2} \int_0^1 \frac{u}{u^2+1} du + \frac{1}{2} \int_0^1 \frac{1}{u^2+1} du.$$

by substitution $2x = u, 2dx = du$, we get

$$\int_0^{\frac{1}{2}} \frac{2x+1}{4x^2+1} dx = \left[\frac{1}{4} \ln(u^2+1) + \frac{1}{2} \arctan u \right]_0^1 =$$

$$= \frac{1}{4} \ln 2 + \frac{1}{2} \arctan 1 = \frac{\ln 2}{4} + \frac{\pi}{8}, \text{ since } \ln 1 = 0, \arctan 0 = 0.$$

For 6)

by parts ($du = \sin \frac{x}{3} dx$. by integr... $u = -3 \cos \frac{x}{3}, v = e^{3x}$. by diff... $dv = 3e^{3x} dx$)

$$F(x) = \int e^{3x} \sin \frac{x}{3} dx = e^{3x} (-3 \cos \frac{x}{3}) - \int (-3) \cos \frac{x}{3} \cdot e^{3x} \cdot 3 dx =$$

$$= -3e^{3x} \cos \frac{x}{3} + 9 \int \cos \frac{x}{3} \cdot e^{3x} dx =$$

by parts again ($du = \cos \frac{x}{3} dx$... $u = 3 \sin \frac{x}{3}, v$..as above)

$$= -3e^{3x} \cos \frac{x}{3} + 9 \left[3 \sin \frac{x}{3} \cdot e^{3x} - \int 3 \sin \frac{x}{3} \cdot 3e^{3x} dx \right] =$$

$$= -3e^{3x} \cos \frac{x}{3} + 27e^{3x} \sin \frac{x}{3} - 81F(x), \text{ solve for } F(x) :$$

$$82 \cdot F(x) = -3e^{3x} \cos \frac{x}{3} + 27e^{3x} \sin \frac{x}{3} + c,$$

$$\text{thus } F(x) = \frac{1}{82} (-3e^{3x} \cos \frac{x}{3} + 27e^{3x} \sin \frac{x}{3}) + c.$$

For 7)

Domain $x \geq 0$ and subst $u = 2 + \sqrt{x} \quad (u-2)^2 = x \quad 2(u-2)du = dx$

$$\int \frac{\sqrt{x}}{2+\sqrt{x}} dx. = \int \frac{u-2}{u} \cdot 2(u-2)du = 2 \int \frac{u^2 - 4u + 4}{u} du =$$

$$\begin{aligned}
&= 2 \int \left(u - 4 + \frac{4}{u} \right) du = u^2 - 8u + 8 \ln |u| = \\
&= (2 + \sqrt{x})^2 - 8(2 + \sqrt{x}) + 8 \ln(2 + \sqrt{x}) + c = \\
&\text{(also)} = x - 4\sqrt{x} + 8 \ln(2 + \sqrt{x}) + c.
\end{aligned}$$

For 8)

use inverse subst. $u = 2 + \sqrt{x+1}$ $(u-2)^2 = x+1$ and $dx = 2(u-2)du$

For $x \geq -1$

$$\begin{aligned}
\int \frac{1}{2 + \sqrt{x+1}} dx &= \int \frac{2(u-2)}{u} du = 2 \int \frac{u-2}{u} du = 2 \int du - 4 \int \frac{1}{u} du = \\
&= 2u - 4 \ln |u| + c = 4 + 2\sqrt{x+1} - 4 \ln |2 + \sqrt{x+1}| + c.
\end{aligned}$$

OR by subst.

$$u = 2 + \sqrt{x+1} \quad du = \frac{1}{2\sqrt{x+1}} dx \text{ and}$$

$$\int \frac{1}{2 + \sqrt{x+1}} \cdot \frac{2\sqrt{x+1}}{2\sqrt{x+1}} dx = \int \frac{2\sqrt{x+1}}{2 + \sqrt{x+1}} \cdot \frac{dx}{2\sqrt{x+1}} = \int \frac{2(u-2)}{u} du = ..$$

as above.

For 9)

$$\int_{-1}^0 x^2 e^{-3x} dx = (\text{by parts} :$$

$$du = e^{-3x} dx \dots \text{by integr} \dots u = -\frac{1}{3} e^{-3x}, v = x^2, \text{by diff} \dots dv = 2x dx)$$

$$= \left[\frac{e^{-3x}}{-3} x^2 \right]_{-1}^0 - \int_{-1}^0 \frac{e^{-3x}}{-3} \cdot 2x dx = 0 + \frac{e^3}{3} + \frac{2}{3} \int_{-1}^0 x e^{-3x} dx$$

(by parts again: $du =$ as above, $v = x \dots dv = 1 dx$)

$$= \frac{e^3}{3} + \frac{2}{3} \left[\frac{e^{-3x}}{-3} x - \int \frac{e^{-3x}}{-3} \cdot 1 dx \right]_{-1}^0 = \frac{e^3}{3} + \frac{2}{3} \left[0 + \frac{-e^3}{3} \right] + \frac{2}{9} \left[\frac{e^{-3x}}{-3} \right]_{-1}^0 =$$

$$= e^3 \left(\frac{1}{3} - \frac{2}{9} \right) - \frac{2}{27} (1 - e^3) = e^3 \frac{9-6+2}{27} - \frac{2}{27} = \frac{5e^3 - 2}{27}.$$

For 10)

Complete the square $x^2 - 4x = (x-2)^2 - 4$ so

$$I = \int_0^2 x \sqrt{4x - x^2} dx = \int_0^2 x \sqrt{4 - (x-2)^2} dx =$$

(using $u = x - 2, x = u + 2, du = dx, u = -2, 0$)

$$= \int_{-2}^0 (u+2) \sqrt{4-u^2} du = \int_{-2}^0 u \sqrt{4-u^2} du + 2 \int_{-2}^0 \sqrt{4-u^2} du =$$

for the first integral use substitution $4 - u^2 = v, u du = -\frac{1}{2} dv,$

for the second use **Table** or by parts or by subst. $u = 2 \sin t, du = 2 \cos t dt$

$$\text{so } I = -\frac{1}{2} \int_0^4 \sqrt{v} dv + \left[u\sqrt{4-u^2} + 4 \arcsin \frac{u}{2} \right]_{-2}^0$$

$$\left[\text{Or } 4 \int_0^{\frac{\pi}{2}} \cos t \sqrt{4-4\sin^2 t} dt = (\dots 8 \cos^2 t) = 4 \int_0^{\frac{\pi}{2}} (1 + \cos 2t) dt \right]$$

$$= \left[-\frac{1}{3} v^{\frac{3}{2}} \right]_0^4 + 4 \arcsin 1 = -\frac{8}{3} + 4 \cdot \frac{\pi}{2} = 2\pi - \frac{8}{3}.$$

For 11)

by parts:

$$du = (3x + 2) dx. \text{ by integr... } u = \frac{3}{2}x^2 + 2x, v = \ln x. \text{ by diff.... } dv = \frac{dx}{x}$$

$$\int (3x + 2) \ln x = \left(\frac{3}{2}x^2 + 2x \right) \ln x - \int \left(\frac{3}{2}x^2 + 2x \right) \cdot \frac{1}{x} dx =$$

$$= \left(\frac{3}{2}x^2 + 2x \right) \ln x - \int \left(\frac{3}{2}x + 2 \right) dx$$

$$= \left(\frac{3}{2}x^2 + 2x \right) \ln x - \frac{3}{4}x^2 - 2x + c \quad \text{for } x > 0.$$