MATH 253
Handout # 5-SOLUTION

A

For 1) $y' + y\sin x = \cos x \sin x \qquad y(\frac{\pi}{2}) = 3.$

the equation is linear $\quad p(x) = \sin x$,$P(x) = -\cos x$ and $\mu = e^{-\cos x}$

multiply by the ntegrating factor μ

$y'e^{-\cos x} + y\sin x e^{-\cos x} = (ye^{-\cos x})' = e^{-\cos x}\cos x \sin x$

so $ye^{-\cos x} = \int e^{-\cos x}\cos x \sin x \; dx + c$

by subst.$u = -\cos x, du = \sin x dx$ the integral$= -\int e^u u \; du = e^u - ue^u$

so

$ye^{-\cos x} = e^{-\cos x}(1 + \cos x) + c$ and multiply by $e^{\cos x}$

$y = 1 + \cos x + ce^{\cos x}$,finally for $x = \frac{\pi}{2} \qquad 3 = 1 + 0 + c$ so $c = 2$

and the solution is $\quad y = 1 + \cos x + 2e^{\cos x}$ for any x.

For 2) $f(x) = \ln(3x - 2)$ and $x_0 = 1$,

first ....$a_0 = f(1) = \ln 1 = 0$, then

using Chain Rule :$f'(x) = \dfrac{3}{3x-2}$...................... $a_1 = f'(1) = 3$

$f''(x) = 3\,(-1)\,(3x-2)^{-2} \cdot 3 = \dfrac{-9}{(3x-2)^2}$..........$a_2 = \frac{f''(1)}{2} = \frac{-9}{2}$

$f'''(x) = -9\,(-2)\,(3x-2)^{-3} \cdot 3 = \dfrac{54}{(3x-2)^3}$.......$a_3 = \frac{f'''(1)}{3!} = \frac{54}{6} = 9$

and

$T_3(x) = 3(x-1) - \frac{9}{2}(x-1)^2 + 9(x-1)^3$

Now to get $\ln 2$ we have to find x : $\quad 3x - 2 = 2$...so $x = \frac{4}{3}$ and

$\ln 2 \doteq T_3(\frac{4}{3}) = 3 \cdot \frac{1}{3} - \frac{9}{2} \cdot \frac{1}{9} + 9 \cdot \frac{1}{27} = 1 - \frac{1}{2} + \frac{1}{3} = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$

$\ln 2 \doteq \frac{5}{6}$ (not a very good approximation)

For 3) $(x+1)y' + (1+y)x^2 = 0$

re-arrange the equation $\quad \dfrac{dy}{dx} = y' = \dfrac{-x^2(1+y)}{x+1}$.......separable

$y = -1$ for all x is also a solution,$y' = 0$;

for $y \neq -1$ separate

$\dfrac{1}{1+y}dy = \dfrac{-x^2}{x+1}dx$ now integrate $\quad \int \dfrac{1}{1+y}dy = \int \dfrac{-x^2}{x+1}dx$

on the right a rational function so long division first or subst. $u = x + 1$

$\int \dfrac{1 - x^2 - 1}{x+1}dx = \int \dfrac{(1-x)(x+1)}{x+1}dx - \int \dfrac{1}{x+1}dx = x - \frac{x^2}{2} - \ln|x+1| + c$

back to equation

$\ln|1+y| = -\frac{x^2}{2} + x - \ln|x+1| + c \qquad y + 1 = \pm e^c \cdot e^{-\frac{x^2}{2}+x} \cdot e^{-\ln|x+1|}$

finally denote $A = \pm e^c$

$y = -1 + \dfrac{Ae^{-\frac{x^2}{2}+x}}{x+1}$ for $x \neq -1$

For 4) $y' = \dfrac{y}{x-y} \quad y(-2) = 1.$

$y' = \dfrac{y}{x-y}$ not sep.,not linear so re-arrange for $x \neq 0, y \neq x$

$y' = \dfrac{\frac{y}{x}}{1-\frac{y}{x}}$ subst.$u = \frac{y}{x}, y' = u'x + u$ gives a separable equation

$u'x + u = \dfrac{u}{1-u}$ $\quad u'x = \dfrac{u}{1-u} - u = \frac{u-u+u^2}{1+u} = \dfrac{u^2}{1-u}$,$u \neq 1$

separate

$\dfrac{1-u}{u^2}du = \dfrac{dx}{x}$ integrate $\int \left(\frac{1}{u^2} - \frac{1}{u}\right) du = \ln|x| + c$

so

$\dfrac{-1}{u} - \ln|u| = \ln|x| + c$ back to y

$(\ln\left|\frac{y}{x}\right| = \ln|y| - \ln|x|)$ thus

$\dfrac{-x}{y} - \ln|y| + \ln|x| = \ln|x| + c$

and cancel $\ln|x|$ and multiply by $y: -x - y\ln|y| = cy$ for any $y \neq 0$

now $x = -2, y = 1$,solve for c

$2 = c$ and together $\quad x = -y\ln|y| - 2y, y \neq 0$ (implicit form)

B

For 1) $\quad f(x) = e^{1-4x^2}$and $x_0 = \frac{1}{2}$,

first ....$a_0 = f(\frac{1}{2}) = e^0 = 1$, then

using Chain Rule :$f'(x) = e^{1-4x^2}(-8x)$ .... $a_1 = f'(\frac{1}{2}) = -4$

Product Rule: $f''(x) = (-8)\, e^{1-4x^2}(1-8x^2)$.........$a_2 = \frac{f''(\frac{1}{2})}{2} = \frac{8}{2} = 4$

and

$T_2(x) = 1 - 4(x - \frac{1}{2}) + 4(x - \frac{1}{2})^2$

Now to estimate $e^{\frac{3}{4}}$ we have to find x such that $1 - 4x^2 = \frac{3}{4}$,

so $\frac{1}{4} = 4x^2$,and $x^2 = \frac{1}{4^2}, x = \pm\frac{1}{4}$ but $\frac{1}{4}$ is closer to $\frac{1}{2}$

$e^{\frac{3}{4}} \doteq T_2(\frac{1}{4}) = 1 - 4 \cdot \frac{-1}{4} + 4 \cdot \frac{1}{16} = 2.25$ (not a very good approximation)

For 2) $\quad y' - 2xy = x$

the equation is linear $\quad p(x) = -2x$,$P(x) = -x^2$ and $\mu = e^{-x^2}$

multiply by the integrating factor μ

$y'e^{-x^2} - 2xye^{-x^2} = \left(ye^{-x^2}\right)' = xe^{-x^2}$ so $ye^{-x^2} = \displaystyle\int xe^{-x^2}dx + c$

by subst. $u = -x^2, du = -2xdx$,integral $= -\dfrac{1}{2}\displaystyle\int e^u du = \dfrac{-1}{2}e^u$

so $\quad ye^{-x^2} = -\dfrac{1}{2}e^{-x^2} + c$ and multiply by $e^{x^2}to$ get

$y = \dfrac{-1}{2} + ce^{x^2}$ for any x.

Note: The equation is also separable.

For 3) $\quad (x^2+1)y' + 2(1+y)x^2 = 0$

re-arrange the equation $\quad \frac{dy}{dx} = y' = \dfrac{-2x^2(1+y)}{x^2+1}$.......sep.

$y = -1$ for any x is also a solution,$y' = 0$;for $y \neq -1$

$\dfrac{1}{1+y}dy = \dfrac{-2x^2}{x^2+1}dx$ now integrate $\quad \int \dfrac{1}{1+y}dy = \int \dfrac{-2x^2}{x^2+1}dx$

$\ln|1+y| = -2\int \frac{x^2+1-1}{x^2+1}dx = -2x + 2\int \frac{1}{x^2+1}dx = -2x + 2\arctan x + c$
$|1+y| = e^{-2x+\arctan x}\cdot e^c \qquad y = -1 + Ae^{-2x+\arctan x}$, where $A = \pm e^c$

For 4) $\qquad y' = \frac{y}{x+y}$
$y' = \frac{y}{x+y}$ not sep.,not linear so re-arrange for $x \neq 0, y \neq -x$
$y' = \frac{\frac{y}{x}}{1+\frac{y}{x}}$ subst.$u = \frac{y}{x}, y' = u'x + u$ gives a separable equation
$u'x + u = \frac{u}{1+u} \qquad u'x = \frac{u}{1+u} - u = \frac{u-u-u^2}{1+u} = -\frac{u^2}{1+u}$
separate for $u \neq 0$ but $u = 0 (y = 0$ for $x \neq 0)$ is also a sol.
$\frac{1+u}{u^2}du = -\frac{dx}{x}$ integrate $\int\left(\frac{1}{u^2}+\frac{1}{u}\right)du = -\ln|x| + c$
so
$\frac{-1}{u} + \ln|u| = -\ln|x| + c$ back to y :using $\ln\left|\frac{y}{x}\right| = \ln|y| - \ln|x|$ we get
$\frac{-x}{y} + \ln|y| - \ln|x| = -\ln|x| + c$ and $y\ln|y| - cy = x$ for any $y \neq 0$
C
For 1) $\qquad f(x) = \arctan(3x)$ and $x_0 = 0$,
first ....$a_0 = f(0) = \arctan 0 = 0$, then
using Chain Rule :$f'(x) = \frac{3}{1+(3x)^2} = \frac{3}{1+9x^2}$........ $a_1 = f'(0) = 3$
$f''(x) = 3(-1)(1+9x^2)^{-2}\cdot 18x = \frac{-54x}{(1+9x^2)^2}$..........$a_2 = \frac{f''(0)}{2} = 0$
by Q.R.
$f'''(x) = (-54)\frac{(1+9x^2)^2 - x\cdot 2(1+9x^2)18x}{(1+9x^2)^4}$..........$a_3 = \frac{f'''(0)}{3!} = \frac{-54}{6} = -9$
and $T_3(x) = 3x - 9x^3$

Now to get $\arctan 1$ we have to substitute for $x = \frac{1}{3}$
$\arctan 1 \doteq T_3(\frac{1}{3}) = 3\cdot\frac{1}{3} - 9\cdot\frac{1}{27} = 1 - \frac{1}{3} = \frac{2}{3}$
$\pi = 4\arctan 1 \doteq \frac{8}{3}$ (not a very good approximation)
For 2) $\qquad xy' = x^2 + y$
for $x \neq 0$ we can re-write the equation as $y' - \frac{y}{x} = x$
so it is linear and $p(x) = \frac{-1}{x}$,$P(x) = -\ln|x| = \ln\left|\frac{1}{x}\right|$ and $\mu = \frac{1}{x}$
multiply by the integrating factor μ
$y'\frac{1}{x} - \frac{y}{x^2} = \left(\frac{y}{x}\right)' = 1 \qquad \frac{y}{x} = \int dx + c = x + c$
and finally $\qquad y = x^2 + cx$ for any x.
For 3) $\qquad (x+y)y' + y - 3x = 0$
re-arrange the equation $\quad y' = \frac{3x-y}{x+y}$.......not sep.,non-linear but homog.
for $y \neq -x, x \neq 0$

$y' = \dfrac{3x-y}{x+y} = \dfrac{3-\frac{y}{x}}{1+\frac{y}{x}}$.Now substitution.....$u = \frac{y}{x}, y' = u'x + u$ gives

$u'x + u = \dfrac{3-u}{1+u}$........$u'x = \dfrac{3-u}{1+u} - u = \dfrac{3-u-u-u^2}{1+u} = \dfrac{3-2u-u^2}{1+u}$......separable

$\dfrac{1+u}{3-2u-u^2}du = \dfrac{dx}{x}$ for integration of the left-hand side

$\displaystyle\int \frac{1+u}{3-2u-u^2}du = -\tfrac{1}{2}\int \frac{2u+2}{u^2+2u-3}du = -\tfrac{1}{2}\ln|u^2+2u-3|$ since the top=(bottom)$'$

\OR

use partial fractions

$\dfrac{1+u}{3-2u-u^2} = -\dfrac{1+u}{u^2+2u-3} = -\dfrac{1+u}{(u-1)(u+3)} = \dfrac{A}{u-1} + \dfrac{B}{u+3}$

so $-(1+u) = A(u+3) + B(u-1)$.......$u = 1$......$-2 = 4A$.....$A = \frac{-1}{2}$

for $u = -3$.........$2 = -4B$......$B = \frac{-1}{2}$....back to the diff.equation

$\frac{-1}{2}\ln|(u+3)(u-1)| = \ln|x| + c$.....$\ln|(u+3)(u-1)| = -2\ln|x| + C$

apply exp.function to both sides

$|(u+3)(u-1)| = e^C \cdot |x|^{-2}$........$(u+3)(u-1) = A\frac{1}{x^2}$....back to y

$\left(\dfrac{y+3x}{x}\right)\left(\dfrac{y-x}{x}\right) = \dfrac{A}{x^2}$.....$(y+3x)(y-x) = A$......general solution

For 4)

$y'(x^2+1)\cos y = -x\sin y$ separable

for $\sin y \neq 0$ $\quad \dfrac{\cos y}{\sin y}dy = \dfrac{-x}{x^2+1}dx$

integrate

$\int \dfrac{\cos y}{\sin y}dy = (\text{subst.}u = \sin y, du = \cos y dy) = \int \frac{du}{u} = \ln|u| = \ln|\sin y|$

and $\displaystyle\int \frac{-xdx}{x^2+1} = (\text{subst.}x^2+1 = u, xdx = \tfrac{1}{2}du) = -\tfrac{1}{2}\ln|x^2+1| + c$

back to diff. equation

$\ln|\sin y| = \ln|x^2+1|^{-\frac{1}{2}} + c$................$\sin y = \pm e^c \cdot (x^2+1)^{-\frac{1}{2}} = \dfrac{A}{\sqrt{x^2+1}}$

to find A substitute $x = 0, y = -\frac{\pi}{2}$

$\sin\frac{-\pi}{2} = -1 = A \cdot 1$ so $A = -1$ finally $\sin y = -\dfrac{1}{\sqrt{x^2+1}}$

and $y = \arcsin\frac{-1}{\sqrt{x^2+1}}$ for any x since $-1 \leq \frac{-1}{\sqrt{x^2+1}} < 0$

D

For 1) $\quad y' - y = e^x \ln x \quad\quad y(1) = -1$

it is linear and $p(x) = -1$,$P(x) = -x$ and $\mu = e^{-x}$

multiply by the integrating factor μ

$y'e^{-x} - ye^{-x} = (ye^{-x})' = e^x \ln x\ e^{-x} = \ln x$ so for $x > 0$

$ye^{-x} = \displaystyle\int \ln x\ dx + c = x\ln x - x + c$,multiply by $e^x$to get

$y = xe^x \ln x - xe^x + ce^x$ for $x > 0$.

for 2)

T_3 for $f(x) = e^{1-x^2}$ and $x_0 = -1$

first ....$a_0 = f(-1) = e^0 = 1$, then

using Chain Rule :$f'(x) = e^{1-x^2}\,(-2x)$ .... $a_1 = f'(-1) = 2$
by Product Rule
$f''(x) = (-2)\,e^{1-x^2}\,[1-2x^2]$ ....$a_2 = \frac{f''(-1)}{2} = \frac{2}{2} = 1$
$f'''(x) = (-2)\,e^{1-x^2}\,[-4x-2x+4x^3]$ ....$a_3 = \frac{f'''(-1)}{3!} = \frac{-4}{6} = \frac{-2}{3}$
and
$T_3(x) = 1 + 2(x+1) + (x+1)^2 - \frac{2}{3}(x+1)^3$

Now to estimate $e^{\frac{3}{4}}$we have to find x such that $1 - x^2 = \frac{3}{4}$
so $x^2 = \frac{1}{4}$ and $x = \pm\frac{1}{2}$ but $x = \frac{-1}{2}$ is *closer* to -1)
$e^{\frac{3}{4}} \doteq T_3(-\frac{1}{2}) = 1 + 2\cdot\frac{1}{2} + \frac{1}{4} - \frac{2}{3}\cdot\frac{1}{8} = 2 + \frac{3-1}{12} = 2 + \frac{1}{6} = 2.16667.$

For 3) $\quad yy' = 2y - x \qquad y(1) = 0$

For $y \neq 0$ rewrite the equation as $y' = 2 - \dfrac{x}{y}$ so it is clearly homog.II type,first order
the substitution $u = \frac{y}{x}$ will give us a separable equation $u'x + u = 2 - \frac{1}{u}$
$u'x = 2 - u - \frac{1}{u} = \dfrac{2u - u^2 - 1}{u}$,separate $\dfrac{-u}{u^2 - 2u + 1}du = \dfrac{dx}{x}$
$\dfrac{-u}{(u-1)^2}du = \dfrac{dx}{x} = \ln|x| + C$ for $u \neq 1$
Now $\displaystyle\int \frac{-u}{(u-1)^2}du = \int \frac{1-u-1}{(u-1)^2}du = \int \frac{-1}{u-1}du - \int (u-1)^{-2}\,du =$
$= -\ln|u-1| + (u-1)^{-1}$ $\qquad$ (or subst. $v = u - 1$
so $\qquad -\ln|u-1| + (u-1)^{-1} = \ln|x| + C,$
back to y :using $\ln|u-1| = \ln|y-x| - \ln|x|$
$\left(\frac{y}{x} - 1\right)^{-1} = \ln|y-x| + C \qquad \dfrac{x}{y-x} = \ln|y-x| + C$ for $y \neq x$
but also ($u = 1$) $y = x$ **is a solution satisfying** $y(0) = 0$.
Now,if $x = 1, y = 0$,so solve for $C : -1 = C$
and the solution is $\qquad \dfrac{x}{y-x} = \ln|y-x| - 1.$

For 4) $\quad x\ln x \cdot y' = y$

the equation is first order,separable,also linear homog.: for $x > 0, x \neq 1$
we can separate : $\dfrac{dy}{y} = \dfrac{dx}{x\ln x}$ and by integrating we get:
$\ln|y| = \int \dfrac{dx}{x\ln x} = \ln|\ln x| + C$ (by subst.$v = \ln x, dv = \frac{dx}{x}$)
$|y| = e^{\ln|\ln x| + C} = e^C \cdot |\ln x| \qquad$ so $\qquad y = K\ln x, x > 0.$