MATH 253
MIDTERM HANDOUT-solution

1. Which of the following are partial fractions? Answer **YES** or **NO**.

   (a)$\dfrac{2}{3-x}$YES (b)$\dfrac{2x+1}{x^3+8}$NO since x^3

   (c)$\dfrac{3x+1}{x^2-4x+5}$YES since No real roots in denom.$D=16-20=-4<0$

   (d)$\dfrac{3x+1}{x^2-4x+3}$NO since 2 real roots in denom.$D=16-12=4>0$

   (e)$\dfrac{3x+1}{(x^2-4x+5)^2}$YES since complex double roots in denom.

   (f)$\dfrac{x^2}{x^2+4}$NO since x^2on the top.

2. Find the inverse function f^{-1}, its domain and range if $f(x)=\arcsin(2x+3)$.

   From the properties of arcsin $\quad -1\le 2x+3\le 1 \quad -4\le 2x\le -2$

   so $D_f=[-2,-1]$ and $R_f=\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

   solve $y=\arcsin(2x+3)$ so $\sin y=2x+3$ and $x=\frac{1}{2}(\sin y-3)$

   and finally

   $f^{-1}(x)=\frac{1}{2}\sin x-\frac{3}{2}$ BUT ONLY for $x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]=D_{f^{-1}}$ and $R_{f^{-1}}=[-2,-1]$

3. Find the domain and antiderivative of $f(x)=\dfrac{\ln(3x)}{x^2}$

   for domain $3x>0$ so $D=(0,\infty)$

   now we can use by parts or inverse subst.

   $$\int x^{-2}\ln(3x)dx=-x^{-1}\ln(3x)+\int x^{-1}\cdot\tfrac{1}{3x}\cdot 3dx=-\frac{\ln 3x}{x}+\int x^{-2}dx=-\frac{\ln 3x}{x}-\frac{1}{x}+c,\quad x>0$$

   OR

   $$u=\ln 3x=\ln 3+\ln x \quad du=\frac{dx}{x} \quad e^u=3x \text{ so } \frac{1}{x}=\frac{3}{e^u}=3e^{-u}$$

   and

   $$\int\frac{\ln 3x}{x^2}dx=\int\frac{\ln 3x}{x}\cdot\frac{dx}{x}=3\int ue^{-u}du=\text{(by parts)}=$$

   $$=-3ue^{-u}+3\int e^{-u}\cdot 1du=-3ue^{-u}-3e^{-u}+c=\text{(back to x)...}$$

4. Find the domain and antiderivative of $f(x)=\dfrac{5x^2+2}{x^3-2x^2+x}$.

   investigate the polynomial in the denominator

   $Q(x)=x^3-2x^2+x=x\left(x^2-2x+1\right)=x(x-1)^2$

   Split the integrand function f into partial fractions

   $$f(x)=\frac{5x^2+2}{x^3-2x^2+x}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{(x-1)^2},$$

   multiply by common denom.: $x(x-1)^2$

   $(*)\quad 5x^2+2=A(x-1)^2+Bx(x-1)+Cx$, substitute $x=0,1,-1$

$2 = A; \quad 7 = C$ and $\quad 7 = 4A + 2B - C$ so $2B = 14 - 8 = 6, B = 3$

Now integrate for $x \neq 0, 1$

$$\int \frac{5x^2+2}{x^3-2x^2+x} dx = 2\int \frac{dx}{x} + 3\int \frac{dx}{x-1} + 7\int (x-1)^{-2}\, dx =$$

$$= 2\ln|x| + 3\ln|x-1| - \frac{7}{x-1} + c = \ln\left(x^2\,|x-1|^3\right) - \frac{7}{x-1} + c, \text{ for } x \neq 0, 1$$

5. for both parts we need

$$F(x) = \int \frac{dx}{x^2-9} \text{ (by Table or Partial fraction)}$$

$$= \tfrac{1}{6}\int \left[\frac{1}{x-3} - \frac{1}{x+3}\right] dx = \tfrac{1}{6}\ln\left|\frac{x-3}{x+3}\right| = \tfrac{1}{6}\ln|x-3| - \tfrac{1}{6}\ln|x+3|$$

for a)

$$\int_4^\infty \frac{dx}{x^2-9} = \lim_{x\to\infty} F(x) - F(4) =$$

$$= \tfrac{1}{6}\ln\left|\lim_{x\to\infty} \tfrac{x-3}{x+3}\right| - \tfrac{1}{6}\ln\tfrac{1}{7} = \tfrac{1}{6}\left[\ln 1 - \ln\tfrac{1}{7}\right] = \tfrac{1}{6}\ln 7 \qquad \text{convergent.}$$

for b)

$$\int_0^3 \frac{dx}{x^2-9} = \lim_{x\to 3^-} F(x) - F(0) = \tfrac{1}{6}\lim_{x\to 3^-} \ln|x-3| - \tfrac{1}{6}\ln 6 - \tfrac{1}{6}\ln 1 = -\infty$$

since " $\ln 0+$ " $= -\infty$, so the integral is divergent.

6. For $\displaystyle\int_0^1 \frac{3}{2+\sqrt{3x+1}} dx$ use inverse subst. $u = 2 + \sqrt{3x+1} \qquad u - 2 = \sqrt{3x+1}$

$(u-2)^2 = 3x+1, \qquad \tfrac{1}{3}\left[(u-2)^2 - 1\right] = x \qquad \tfrac{2}{3}(u-2)\,du = dx,$

x	u
0	3
1	4

$$\int_0^1 \frac{3}{2+\sqrt{3x+1}} dx = 2\int_3^4 \frac{u-2}{u} du = 2\int_3^4 du - 4\int_3^4 \frac{du}{u} = 2 - 4\ln\tfrac{4}{3}.$$