Lab Quiz 4 Solutions, Math 253 B25/B26 Wed. 03/23 , 1300 hours

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1. Explain why each integral is improper. Decide whether convergent or divergent (and justify your conclusion). If convergent, evaluate:

$$\text{(a)} \int_1^4 \frac{x}{x^2-1}\,dx \quad \text{(b)} \int_1^\infty \frac{1}{x^2+1}\,dx\ .$$

Solution: (a) The integral is improper because the integrand has a vertical asymptote (blows up) at $x = 1$. We check for convergence by trying to evaluate:

$$\lim_{A\to 1+} \int_A^4 \frac{x}{x^2-1}\,dx = \lim_{A\to 1+} \left[\frac{1}{2}\ln\left(x^2-1\right)|_A^4\right] =$$

$$= \frac{1}{2}\lim_{A\to 1+}\left[\ln(15) - \ln\left(A^2-1\right)\right] == +\infty.$$

Divergent.

(b) The integral is improper because the interval of integration is infinite. We evaluate as a limit:

$$\lim_{T\to\infty} \int_1^T \frac{1}{x^2+1}\,dx == \lim_{T\to\infty}\left[\arctan(T) - \frac{\pi}{4}\right] = \frac{\pi}{4}.$$

Converges.

2. Use comparison to decide if the integral converges or diverges:

$$\int_1^\infty \frac{\arctan(x)}{x^{\frac{3}{2}}}\,dx.$$

Solution: We know that for $1 \le x \le \infty$,

$$\frac{\pi}{4} \le \arctan(x) < \frac{\pi}{2}\ .$$

This in turn implies that

$$\frac{\frac{\pi}{4}}{x^{\frac{3}{2}}} \le \frac{\arctan(x)}{x^{\frac{3}{2}}} < \frac{\frac{\pi}{2}}{x^{\frac{3}{2}}}\ .$$

Now $\int_1 \infty \frac{\frac{\pi}{2}}{x^{\frac{3}{2}}}\,dx$ converges since it is just $\frac{\pi}{2}$ times $\int_1^\infty \frac{1}{x^p}\,dx$ with $p > 1$. Therefore the original integral converges.

3. Use the trapezoidal rule with n=3 to approximate the following integral:

$$\int_0^1 \frac{1}{1+x^2}\,dx.$$

Evaluate the integral exactly and compare with your approximation.

Solution: The exact value is $\arctan(1) = \frac{\pi}{4} = 0.7853981635$. The Trapezoidal Rule applied to this integral for $n = 3$ and $a = 0,\ b = 3$ is

$$\frac{b-a}{3}\left[\frac{1}{2}f(0) + f\left(\frac{1}{3}\right) + f\left(\frac{2}{3}\right) + \frac{1}{2}f(1)\right] =$$

$$= \frac{1}{3}\left[\frac{1}{2} + \frac{9}{10} + \frac{9}{13} + \frac{1}{4}\right] = 0.7807692310.$$