Solutions to Lab Quiz 4, Math 253 B27/B28 Friday 04/01, 10:00

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1. Explain why the integrals are improper, and decide whether convergent or divergent. If convergent, evaluate:

$$\text{(a)} \int_1^2 \frac{x^2}{x^3-8}\,dx \quad \text{(b)} \int_1^\infty \frac{1}{x\left[1+\ln^2(x)\right]}\,dx.$$

Solution: (a) Improper because the integrand has a vertical asymptote at $x=2$. We try to evaluate:

$$= \lim_{T\to 2-} \frac{1}{3}\int_1^T \frac{3x^2}{x^3-8}\,dx = \lim_{T\to 2-} \frac{1}{3}\ln|x^3-8|\,|_1^T =$$

$$= \frac{1}{3}\lim_{T\to 2-}\left[\ln|T^3-8| - \ln|1^3-8|\right] = -\infty.$$

The integral diverges.

(b) The integral is improper because the interval of integration is infinite. We try to evaluate:

$$= \lim_{T\to\infty}\int_1^T \frac{1}{x\left[1+\ln^2(x)\right]}\,dx\,|_{\substack{u=\ln(x)\\ du=1/x}} = \lim_{T\to\infty}\int_0^{\ln(T)} \frac{1}{[1+u^2(x)]}\,du =$$

$$= \lim_{T\to\infty}\left[\arctan(\ln(T)) - \arctan(0)\right] = \frac{\pi}{2}.$$

The integral converges.

2. Explain why the integral is improper and use comparison to decide if it is convergent or divergent:

$$\int_1^\infty \frac{2+\sin(x)}{x^{\frac{1}{2}}}\,dx.$$

Solution: The integral is improper because the interval of integration is infinite. We suspect that the integral diverges since it has a denominator ($x^{\frac{1}{2}}$) which indicates divergence. So we should try to bound it below by something divergent:

$$-1 \le \sin(x) \quad\Rightarrow\quad 1 \le 2+\sin(x) \quad\Rightarrow\quad \frac{1}{x^{\frac{1}{2}}} \le \frac{2+\sin(x)}{x^{\frac{1}{2}}}.$$

Since $\int_1^\infty \frac{1}{x^{\frac{1}{2}}}$ diverges, so does the original integral, by comparison.

3. Use the trapezoidal rule with n=3 to approximate the integral

$$\int_0^2 \frac{x}{1+x^2}\,dx.$$

Evaluate the integral exactly and compare the results.

Solution:

$$T_3 = \frac{2-0}{3}\left[\frac{1}{2}f(0) + f\left(\frac{2}{3}\right) + f\left(\frac{4}{3}\right) + \frac{1}{2}f(2)\right] =$$

$$= \frac{2}{3}\left[\frac{1}{2}\cdot 0 + \frac{6}{13} + \frac{12}{25} + \frac{1}{2}\cdot\frac{2}{5}\right] = 0.761$$

The exact value is

$$\frac{1}{2}\ln(1+x^2)\,|_0^2 = \frac{1}{2}\ln(5) = .8047$$

The trapezoidal rule is too low by approximately .043.