Lab Worksheet 1 Solutions, Math 253 B25-32, January 25-29, 2005

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1. For $f(x) = \ln(5x-2)$, show f is one-to one. Find a formula for the inverse function $f^{-1}(x)$, as well as its domain and range.

   Solution: The domain of f is $(\frac{2}{5}, \infty)$, its range (from sketching $\ln(5)+\ln\left(x-\frac{2}{5}\right)$, which is just $y = ln(x)$ shifted vertically by $\ln(5)$ and $\frac{2}{5}$ to the right) is $(-\infty, +\infty) = R$. Thus the domain of f^{-1} is R and the range of f^{-1} is $(\frac{2}{5}, \infty)$. The function is one-to-one by the horizontal line test applied to the graph, or by observing that

   $$f'(x) = \frac{5}{5x-2} > 0$$

   on the domain of f, so f is strictly increasing. To find a formula for the inverse, solve the equation $y = \ln(5x-2)$ for x as a function of y:

   $$e^y = 5x - 2 \;\Rightarrow\; x = \frac{e^y+2}{5},$$

   then interchange x and y to get

   $$f^{-1}(x) = \frac{e^x+2}{5}.$$

2. Simplify each of the following:
   (a) $\tan\left(\arcsin\left(\frac{1}{3}\right)\right)$, (b) $\sin(\arctan(p+1))$.

   Solution: (a) If $\theta = arcsin\left(\frac{1}{3}\right)$ then we can draw a right triangle for θ From the triangle we can see that the tangent of θ is $\frac{1}{\sqrt{8}}$.
   (b) is similar, a simple sketch of the relevant right triangle gives the sine as $\frac{p+1}{(p+1)^2+1}$.

3. All you know about $f(x)$ is that it is differentiable everywhere, and the following values:

   $$f(2) = 10, f(6) = 2, f(10) = 6, f'(2) = 2, f'(6) = 1, f'(10) = -3.$$

   What is the derivative of f^{-1} at 6 ? What is the derivative of $g(x) = \left[f^{-1}(x)\right]^2$ at 6? (Hint for last: Chain Rule)

   Solution: The derivative of f^{-1} at 6 is just the reciprocal of $f'(x)$ at $x = 10$, since 10 is the pre-image of 6. Thus the derivative is $-\frac{1}{3}$. The derivative of $\left[f^{-1}(x)\right]^2$ is, by the chain rule $2f^{-1}(x)f^{-1\prime}(x)$ which at $x = 10$ is $2 \cdot 10 \cdot (-\frac{1}{3})$.

4. Evaluate the following integrals:

   $$\text{(a)} \int_0^{\frac{\pi}{2}} \frac{\cos(x)}{1+\sin^2(x)}\,dx \quad \text{(b)} \int_1^2 \frac{1}{[2x-x^2]^{\frac{1}{2}}}\,dx.$$

   Simplify your answer as much as possible.

Solution: (a) Use the substitution $u = \sin(x)$ to get

$$\int_0^1 \frac{1}{1+u^2}\,du = \arctan(u)|_0^1 = \frac{\pi}{4}.$$

(b) Complete the square on the bottom to get

$$\int_1^2 \frac{1}{[1-(x-1)^2]^{\frac{1}{2}}}\,dx = \arcsin(x-1)|_1^2 =$$

$$= \arcsin(1) - \arcsin(0) = \frac{\pi}{2}.$$