Mathematics 253 Lab Worksheet, Labs B25-B32

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1. Use partial fractions to evaluate the following (remember to divide if the degree of the numerator exceeds the degree of the denominator):

$$\text{(a)} \int \frac{3x^2-7x-4}{x^3-2x^2-x+2}\,dx, \quad \text{(b)} \int \frac{-x^2+11x}{(x-1)(x^2+4)}\,dx, \quad \text{(c)} \int \frac{x^4}{x^2-1}\,dx.$$

Hint: In (a) the denominator vanishes for $x = 1$.

Solution: (a) The polynomial x^3-2x^2-x+2 vanishes at $x = 1$, so $(x-1)$ is a factor. We can then factor the polynomial: $x^3-2x^2-x+2 = (x-1)(x^2-x-2)$, and the quadratic factors easily to give

$$\frac{3x^2-7x-4}{x^3-2x^2-x+2} = \frac{3x^2-7x-4}{(x-1)(x-2)(x+1)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x+1}.$$

The theory of partial fractions tells us that this will be solvable for A, B, C. We re-assemble the right side over a common denominator and equate the resulting numerator to the original numerator:

$$3x^2-7x-4 = A(x-2)(x+1) + B(x-1)(x+1) + C(x-1)(x-2).$$

If we set $x = -1$ in this identity we get $3+7-4 = 6C \Rightarrow C = -1$. If we set $x = 2$, we get $12-14-4 = 3B \Rightarrow B = -2$. If we set $x = 1$ we get $3-7-4 = -2A \Rightarrow A = 4$. Therefore

$$\int \frac{3x^2-7x-4}{x^3-2x^2-x+2}\,dx = \int \left[\frac{4}{x-1} - \frac{2}{x-2} - \frac{1}{x+1}\right] dx =$$

$$= 4\ln|x-1| - 2\ln|x-2| - \ln|x+1| + C = \ln\left[\frac{(x-1)^4}{(x-2)^2|x+1|}\right] + C.$$

(b) We know that the integrand can be expanded in partial fractions in the following form:

$$\frac{-x^2+11x}{(x-1)(x^2+4)} = \frac{Ax+B}{x^2+4} + \frac{C}{x-1}.$$

We re-assemble the right side over a common denominator and equate the resulting numerator to the original numerator: $-x^2+11x = (Ax+B)(x-1)+C(x^2+4)$. Setting x = 1, we get $-1+11 = 5C \Rightarrow C = 2$. To get the remaining unknowns we have to resort to equating the coefficients of powers of x:

$$x^2: \quad -1 = A + C \Rightarrow A = -3;$$

$$x: 11 = -A + B \Rightarrow B = 8.$$

Thus

$$\int \frac{-x^2+11x}{(x-1)(x^2+4)}\,dx = \int \left[\frac{-3x+8}{x^2+4} + \frac{2}{x-1}\right] dx =$$

$$= -\frac{3}{2}\int \frac{2x}{x^2+4}\,dx + \frac{8}{4}\int \frac{1}{1+\left[\frac{x}{2}\right]^2}\,dx + 2\ln|x-1| + C =$$

$$= -\frac{3}{2}\ln\left(x^2+4\right) + 4\arctan\left(\frac{x}{2}\right) + 2\ln|x-1| + C.$$

(c) We divide the polynomials to get:

$$\frac{x^4}{x^2-1} = x^2 + 1 + \frac{1}{x^2-1}.$$

By partial fractions, we have

$$\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)} = \frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1},$$

so

$$\int \frac{x^4}{x^2-1}\,dx = \int \left[x^2 + 1 + \frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1}\right]\,dx =$$

$$= \frac{1}{3}x^3 + x + \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C = \frac{1}{3}x^3 + x + \ln\left[\frac{\sqrt{|x-1|}}{\sqrt{|x+1|}}\right] + C.$$

2. Use integration by parts ($U = \sin^{n-1}(x)$) to show that

$$I_n = \int \sin^n(x)\,dx = -\cos(x)\sin^{n-1}(x) + (n-1)\int \cos^2(x)\sin^{n-2}(x)\,dx$$

then use $\cos^2(x) = 1 - \sin^2(x)$ to get the recursion formula

$$I_n = -\frac{1}{n}\cos(x)\sin^{n-1}(x) + \frac{n-1}{n}I_{n-2}.$$

Now use the fact that $\int_0^{\frac{\pi}{2}} \sin(x) = 1$ and the above reduction formula to show that

$$\int_0^{\frac{\pi}{2}} \sin^3(x)\,dx = \frac{2}{3}, \text{ and } \int_0^{\frac{\pi}{2}} \sin^5(x)\,dx = \frac{4\cdot 2}{5\cdot 3}.$$

Look at the pattern and predict the value of $\int_0^{\frac{\pi}{2}} \sin^{11}(x)\,dx$.

Solution: We set

$$U = \sin^{n-1}(x), \quad dV = \sin(x)\,dx$$
$$dU = (n-1)\sin^{n-2}(x)\cos(x)\,dx, \quad V = -\cos(x)$$

to get

$$I_n = \int \sin^n(x)\,dx = -\cos(x)\sin^{n-1}(x) + (n-1)\int \sin^{n-2}(x)\cos^2(x)\,dx =$$

$$= -\cos(x)\sin^{n-1}(x) + (n-1)\int \sin^{n-2}(x)\left[1-\sin^2(x)\right]\,dx =$$

$$= -\cos(x)\sin^{n-1}(x) + (n-1)I_{n-2} - (n-1)I_n.$$

Thus we have $I_n = -\cos(x)\sin^{n-1}(x) + (n-1)I_{n-2} - (n-1)I_n$ and we can solve for I_n to get

$$I_n = -\frac{1}{n}\cos(x)\sin^{n-1}(x) + \frac{n-1}{n}I_{n-2}.$$

If we have limits of 0 and $\frac{\pi}{2}$ on the integral then we have

$$I_n = -\frac{1}{n}\cos(x)\sin^{n-1}(x)|_0^{\frac{\pi}{2}} + \frac{n-1}{n}I_{n-2} = \frac{n-1}{n}I_{n-2}.$$

For n=1 we have $I_1 = \int_0^{\frac{\pi}{2}} \sin(x)\,dx = 1$, and for n = 3 we have

$$I_3 = \int_0^{\frac{\pi}{2}} \sin^3(x)\,dx = \frac{3-1}{3}I_{3-2} = \frac{2}{3}.$$

For n = 5 we have

$$I_5 = \int_0^{\frac{\pi}{2}} \sin^5(x)\,dx = \frac{5-1}{5}I_{5-2} = \frac{4}{5}\cdot\frac{2}{3}.$$

This pattern suggests (correctly) that

$$\int_0^{\frac{\pi}{2}} \sin^{11}(x)\,dx = \frac{10\cdot 8\cdot 6\cdot 4\cdot 2}{11\cdot 9\cdot 7\cdot 5\cdot 3}.$$

3. (for those who finish 1 and 2 early): Evaluate (Hint: Kill the ugly function):

$$\int \frac{\arcsin(x)}{\sqrt{1+x}}\,dx.$$

Solution: We integrate by parts:

$$U = \arcsin(x), \quad dV = \frac{1}{\sqrt{1+x}}\,dx$$

$$dU\frac{1}{\sqrt{1-x^2}}\,dx, \quad V = 2\sqrt{1+x}.$$

Then

$$\int \frac{\arcsin(x)}{\sqrt{1+x}}\,dx = \sqrt{1+x}\arcsin(x) - 2\int \frac{\sqrt{1+x}}{\sqrt{1-x^2}}\,dx =$$

$$= \sqrt{1+x}\arcsin(x) - 2\int \frac{1}{\sqrt{1-x}}\,dx = \sqrt{1+x}\arcsin(x) + 4\sqrt{1-x} + C.$$

Here we used the fact that

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}, \text{ with } a = 1-x,\ b = 1-x^2,$$

and also the fact that $1 - x^2 = (1-x)(1+x)$.