MATHEMATICS 253 L08 MIDTERM SOLUTIONS

March 10, 2005

1. Evaluate $\int x^2 \arctan(x)\, dx$. Show all work, and state your answer in terms of the original variable of integration.

   Solution: We integrate by parts, setting

   $$U = \arctan(x),\ dV = x^2\, dx,\ dU = \frac{1}{1+x^2}\, dx,\ V = \frac{1}{3}x^3.$$

   Then the integral is transformed to

   $$= \frac{1}{3}x^3 \arctan(x) - \frac{1}{3}\int \frac{x^3}{1+x^2}\, dx.$$

   The remaining integral can be evaluated using either trig substitution or division of polynomials. Using trig substitution, we set $x = \tan(u),\ dx = \sec^2(u)\, du$ to get

   $$= -\frac{1}{3}\int \frac{\tan^3(u)}{\sec^2(u)} \sec^2(u)\, du = -\frac{1}{3}\int \tan^3(u)\, du =$$

   $$= -\frac{1}{3}\int \left[\sec^2(u) - 1\right] \tan(u)\, du = -\frac{1}{3}\left[\frac{1}{2}\tan^2(u) - \ln|\sec(u)|\right] + C =$$

   $$= -\frac{1}{3}\left[\frac{1}{2}x^2 - \ln|\sqrt{x^2+1}|\right] + C.$$

   The final answer is:

   $$\frac{1}{3}\{x^3 \arctan(x) - \frac{1}{2}x^2 - \ln(\sqrt{x^2+1}|) + C.$$

2. Evaluate

   $$\int \frac{x+1}{[x^2+6x+10]^{\frac{3}{2}}}\, dx$$

   Show all work, and state your answer in terms of the original variable of integration.

   Solution: Complete the square in denominator to get

   $$= \int \frac{x+1}{[(x+3)^2+1]^{\frac{3}{2}}}\, dx,$$

   then set $x + 3 = \tan(u),\ dx = \sec^2(u)\, du$, to get

   $$= \int \frac{\tan(u) - 2}{\sec^3(u)} \sec^2(u)\, du = \int [\sin(u) - 2\cos(u)]\, du =$$

   $$= -\cos(u) - 2\sin(u) + C = \frac{1 - 2(x+3)}{\sqrt{x^2+6x+10}} + C.$$

3. Consider the function

$$f(x) = \ln\left(\frac{\pi}{2} - \arcsin(x)\right).$$

(a) Describe the domain and range of f. (b) Show that f is one-to-one, and describe the domain and range of f^{-1}, the inverse of f. (c) Find a formula for the inverse of f.

Solution: (a) The domain is $[-1, +1)$, only the point $+1$ from the domain of $\arcsin(x)$ is excluded, because at that point $\arcsin(x) = \frac{\pi}{2}$ and then the argument of ln is 0 which is not allowed. The range is the range of $\ln(u)$, for $0 < u \leq \pi$, i.e., $(-\infty, \ln(\pi)]$.

(b) We calculate the derivative of $f(x)$:

$$f'(x) = \frac{1}{\frac{\pi}{2} - \arcsin(x)} \cdot \frac{-1}{\sqrt{1-x^2}},$$

which is never zero, so by Rolles Theorem the function is one-to-one. Domain of f^{-1} is the Range of f and the Range of f^{-1} is the Domain of f, as described in (a).

(c) We have

$$y = \ln\left(\frac{\pi}{2} - \arcsin(x)\right) \Rightarrow e^y = \frac{\pi}{2} - \arcsin(x) \Rightarrow \arcsin(x) = \frac{\pi}{2} - e^y \Rightarrow$$

$$x = \sin\left(\frac{\pi}{2} - e^y\right),$$

so

$$f^{-1}(x) = \sin\left(\frac{\pi}{2} - e^x\right).$$

4. Evaluate

$$\int \frac{3x^2 + 6x - 2}{(x+3)(x^2+x+1)}\, dx \qquad .$$

Show all work, and state your answer in terms of the original variable of integration.

Solution: We use partial fractions:

$$\frac{3x^2 + 6x - 2}{(x+3)(x^2+x+1)} = \frac{A}{x+3} + \frac{Bx + C}{x^2+x+1} \Rightarrow$$

$$3x^2 + 6x - 2 = A(x^2 + x + 1) + (Bx + C)(x + 3).$$

Setting $x = -3$, we get $7 = 7A$, $A = 1$. Putting this into the equation above we have

$$3x^2 + 6x - 2 = x^2 + x + 1 + (Bx + C)(x + 3) \Rightarrow 2x^2 + 5x - 3 = Bx^2 + (3B + C)x + 3C,$$

which gives $B = 2$, $C = -1$. Thus our original integral becomes

$$= \int \frac{1}{x+3}\,dx + \int \frac{2x-1}{x^2+x+1}\,dx =$$

$$= \ln|x+3| + \int \frac{2x+1}{x^2+x+1}\,dx - \int \frac{2}{x^2+x+1}\,dx =$$

$$= \ln|x+3| + \ln|x^2+x+1| - \int \frac{2}{(x+\frac{1}{2})^2+\frac{3}{4}}\,dx =$$

$$= \ln|x+3| + \ln|x^2+x+1| - \frac{4}{\sqrt{3}} \arctan\left[\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right] + C.$$

5. Evaluate

$$\int \frac{1}{x+2\sqrt{x-2}}\,dx.$$

Show all work, and state your answer in terms of the original variable of integration.

Solution: Let $u^2 = x-2$, $2u\,du = dx$, then the integral transforms into

$$= \int \frac{1}{u^2+2+2u} 2u\,du = \int \frac{2u+2-2}{u^2+2u+2}\,du =$$

$$= \int \frac{2u+2}{u^2+2u+2}\,du - 2\int \frac{1}{u^2+2u+2}\,du = \ln|u^2+2u+2| - 2\int \frac{1}{(u+1)^2+1}\,du =$$

$$\ln|u^2+2u+2| - 2\arctan(u+1) + C = \ln(x+2\sqrt{x-2}) - 2\arctan(\sqrt{x-2}+1) + C.$$

6. Consider the bounded region determined by the curves $y = \arcsin(x)$, $y = \frac{\pi}{2}x$, $0 \leq x \leq 1$. Write down the integral for the area of this region as a definite integral (a) on the x-axis, (b) on the y-axis. DO NOT EVALUATE, but be sure to put in limits. (Note: On the given interval, the graph of the line lies above the graph of $\arcsin(x)$.)

Solution:

(a) $$\int_0^1 \left[\frac{\pi}{2}x - \arcsin(x)\right]\,dx.$$

(b) $$\int_0^{\frac{\pi}{2}} \left[\sin(y) - \frac{2}{\pi}y\right]\,dy.$$