The University of Calgary
Faculty of Science
Department of Mathematics and Statistics
MATH 253 (L60) Midterm Test

Time:120 minutes **Summer 2000**

[6] 1. Which of the following are partial fractions? Answer **YES** or **NO**.

a) $\dfrac{2x}{3+x}$ ☐ d) $\dfrac{3x+1}{x^2-4x+7}$ ☐

b) $\dfrac{2x-1}{x^2+8}$ ☐ e) $\dfrac{3}{(x+5)^2}$ ☐

c) $\dfrac{3x}{x^2-4x-7}$ ☐ f) $\dfrac{x^2}{x^2+4x}$ ☐

[6] 2. Giventhe following information on f :
it is one-to-one,continuous and differentiable on $D_f = [3, 7]$,
$R_f = [2, 10]$, $f(4) = 9, f(5) = 4, f'(4) = -\frac{1}{3}, f'(5) = -\frac{1}{6}$
find the domain,range of inverse function , the values $f^{-1}(4)$, $f^{-1}(9)$,
and the derivative of the inverse $(f^{-1})'(4)$.

[8] 3. Evaluate $\int_1^3 \frac{1}{x^3} \sin \frac{\pi}{x} dx$.

[10] 4. Find the domain and antiderivative of $f(x) = \dfrac{2}{x^4 - 4x^2}$.

[10] 5. Find the domain and antiderivative of $f(x) = \dfrac{x^3+1}{x^3+4x^2+8x}$.

[7] 6. Find $\int \cos \sqrt[3]{x}\, dx$.

[6] 7. Is the integral $\int_{-\infty}^{2} e^{3x} dx$ convergent or divergent? If convergent,find the value.

[7] 8. Is the integral $\int_{-1}^{+\infty} \frac{1}{(x+1)^3} dx$ convergent or divergent? If convergent,find the value.

SOLUTION

For 1)

a) NO since x is on the top; b)Yes; c) NO since $D > 0$, two real roots;

d)Yes, $D<0$; e) Yes; f) NO x^2on top.

For 2)

$D_{f^{-1}}=R_f=[2,10]$ and $R_{f^{-1}}=D_f=[3,7]$, $f^{-1}(4)=5,\ f^{-1}(9)=4,$

and $\left(f^{-1}\right)'(4)=\dfrac{1}{f'(5)}=-6$ since $f(5)=4$.

For 3)

$$\int_1^3 \frac{1}{x^3}\sin\frac{\pi}{x}dx=\left(\text{subst. } u=\frac{\pi}{x}, x=\frac{\pi}{u}, dx=-\frac{\pi}{u^2}du, u=\pi,\tfrac{\pi}{3}\right)=$$

$$=-\int_{\pi}^{\frac{\pi}{3}}\left(\frac{u}{\pi}\right)^3\sin u\cdot\tfrac{\pi}{u^2}du=\tfrac{1}{\pi^2}\int_{\frac{\pi}{3}}^{\pi}u\sin u\ du=\tfrac{1}{\pi^2}\{[u(-\cos u)]_{\frac{\pi}{3}}^{\pi}+\int_{\frac{\pi}{3}}^{\pi}\cos u\ du\}$$

$$=\tfrac{1}{\pi^2}\{-\pi\cos\pi+\tfrac{\pi}{3}\cos\tfrac{\pi}{3}+\sin\pi-\sin\tfrac{\pi}{3}\}=\frac{1}{\pi}\left[1+\tfrac{1}{6}\right]-\frac{\sqrt{3}}{2\pi^2}=\frac{7}{6\pi}-\frac{\sqrt{3}}{2\pi^2}.$$

For 4) for $x\neq 0,\pm2$

$$\frac{2}{x^4-4x^2}=\frac{2}{x^2(x^2-4)}=\frac{2}{x^2(x-2)(x+2)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-2}+\frac{D}{x+2}$$

multiply by common denom. $x^2(x-2)(x+2)$

then $(*)$ $2=Ax(x^2-4)+B(x^2-4)+Cx^2(x+2)+Dx^2(x-2)$

for $x=0$ $2=-4B$ $B=\frac{-1}{2}$

for $x=2$ $2=16C$ $C=\frac{1}{8}$

for $x=-2$ $2=-16D$ $D=\frac{-1}{8}$

for e.g. $x=1$ $2=-3A-3B+3C-D=-3A+\frac{3}{2}+\frac{3}{8}+\frac{1}{8}$

$3A=0$ $A=0$

now integration for $x\neq0, x\neq\pm2$

$$\int\frac{2}{x^4-4x^2}dx=\tfrac{-1}{2}\int\frac{dx}{x^2}+\tfrac{1}{8}\int\frac{dx}{x-2}-\tfrac{1}{8}\int\frac{dx}{x+2}=\frac{1}{2x}+\tfrac{1}{8}\ln\left|\frac{x-2}{x+2}\right|+c.$$

For 5)

long division first

$$\frac{x^3+4}{x^3+4x^2+8x}=\frac{(x^3+4x^2+8x)-4x^2-8x+4}{x^3+4x^2+8x}=1-\frac{4x^2+8x-4}{x(x^2+4x+8)}$$

$\dfrac{4(x^2+2x-1)}{x\left(x^2+4x+8\right)}=\dfrac{A}{x}+\dfrac{Bx+C}{x^2+4x+8}$ since denom. has complex roots

multiply by common denom. $x\left(x^2+4x+8\right)$

$(*)$ $4\left(x^2+2x-1\right)=A(x^2+4x+8)+Bx^2+Cx$

for $x=0$ $-4=8A$ $A=\frac{-1}{2}$

for $x=1$ $8=13A+B+C$

for $x=-1$ $-8=5A+B-C$ add two equations to get

$0=18A+2B$ so $B=-9A=\frac{9}{2}$

subtract to get $16=8A+2C$ $C=8-4\cdot\frac{-1}{2}=10$

now integration for $x\neq0$

$$\int \frac{x^3+4}{x^3+4x^2+8x}dx = \int dx - \int \frac{4(x^2+2x-1)}{x\,(x^2+4x+8)}dx =$$

$$= x + \tfrac{1}{2}\int \frac{dx}{x} - \tfrac{1}{2}\int \frac{9x+20}{x^2+4x+8}dx = x + \tfrac{1}{2}\ln|x| - \tfrac{1}{2}\int \frac{9(x+2)+2}{(x+2)^2+4}dx =$$

$$= x + \tfrac{1}{2}\ln|x| - \tfrac{9}{4}\int \frac{2u}{u^2+4}du - \int \frac{du}{u^2+4} = (\text{subst. } u = x+2, x = u-2, dx = du)$$

$$= x + \tfrac{1}{2}\ln|x| - \tfrac{9}{4}\ln\left(x^2+4x+8\right) - \tfrac{1}{2}\arctan\left(\tfrac{x+2}{2}\right) + c.$$

For 6)

for any x

first use subst. $u = \sqrt[3]{x}, u^3 = x, 3u^2du = dx$, then by parts twice

$$\int \cos\sqrt[3]{x}\ dx = 3\int u^2\cos u\ du = (\text{integrate trigs})$$

$$= 3u^2\sin u - 3\int \sin u\cdot 2u\ du = 3u^2\sin u + 6\left[u\cos u - \int \cos u du\right] =$$

$$= 3u^2\sin u + 6u\cos u - 6\sin u = (\text{back to x})$$

$$= 3x^{\frac{2}{3}}\sin\sqrt[3]{x} + 6\sqrt[3]{x}\cos\sqrt[3]{x} - 6\sin\sqrt[3]{x} + c.$$

For 7)

first find $F(x) = \int e^{3x}dx = \frac{1}{3}e^{3x}$ then $\int\limits_{-\infty}^{2} e^{3x}dx = F(2) - \lim\limits_{x\to-\infty} F(x) =$

$= \frac{e^6}{2} - \frac{1}{3}\lim\limits_{x\to-\infty} e^{3x} = \frac{e^6}{3}$ the integral is convergent since the limit is 0.

For 8)

$$\int\limits_{-1}^{+\infty} \frac{1}{(x+1)^3}dx = \lim_{x\to+\infty} F(x) - \lim_{x\to-1^+} F(x)$$

where $F(x) = \int (x+1)^{-3}\,dx = \dfrac{-1}{2(x+1)^2}$

now $\lim\limits_{x\to+\infty}\dfrac{-1}{2(x+1)^2} = \text{"}\frac{1}{\infty}\text{"} = 0$ but $\lim\limits_{x\to-1^+}\dfrac{-1}{2(x+1)^2} = \text{"}\frac{-1}{0^+}\text{"} = -\infty$

thus the integral is divergent.