Solutions to Final Exam Review Questions, Math 253 L07/L08

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Note: I have only proofed these lightly, so there could very well be numerical or other computational errors. However, the general approach will always be correct! If you inform me of errors, I will send you an email to confirm, but there will not be time to post a corrected copy.

1. Domain and range of a function, proving a function is one-to-one, domain and range and derivative of inverse function, formula for inverse function.

   Typical Problems:

   (a) If $f(x) = e^{\sqrt{4-x}}$, find the domain and range of f. Show f is one-to-one on its domain. Find a formula for the inverse function f^{-1}.(Handout 1)
   Solution: See solutions for Handout 1.

   (b) Same as previous problem, for $f(x) = \ln\left[\frac{1}{1-x}\right]$. (Handout 1)

   (c) Same as above for $f(x) = \arcsin(\sqrt{3+x} - 2)$
   Solution: Domain is $[-2,\ 6]$, range is $-1 \le x \le +1$. We calculate the derivative:

   $$f'(x) = \frac{1}{\left[1 - \left(\sqrt{3+x} - 2\right)^2\right]^{\frac{1}{2}}} \frac{1}{2\sqrt{3+x}}.$$

   This is always positive on the domain of f, so the function is strictly increasing, hence one-to-one.
   The domain of f^{-1} is the range of f, and the range of f^{-1} is the domain of f. We find the formula for f^{-1}:

   $$y = \arcsin(\sqrt{3+x} - 2) \quad \Rightarrow \quad \sin(y) = \sqrt{3+x} - 2 \quad \Rightarrow \quad x = -3 + [2 + \sin(y)]^2.$$

   Therefore, $f^{-1}(x) = x = -3 + [2 + \sin(x)]^2$.

2. Integration by parts. Three topics: Kill a power of x, kill the ugly function, find a recursion formula (reduction formula) for a set of integrals. (Problems from handouts have solutions posted already).

   (a) Evaluate $\int x\,(\ln(x))^2\,dx$. (Handout 3)

   (b) Evaluate without Table: $\int_0^{\frac{1}{2}} \arctan(2x)\,dx$. (Handout 3)

   (c) Evaluate $\int_{-1}^{0} x^2 e^{3x}\,dx$ (Handout 3).

   (d) Given $\int x^n e^{x^2}\,dx$, $n = 0, 1, 2, \ldots$, call these integrals I_n, e.g., $I_3 = \int x^3 e^{x^2}\,dx$. Show that

   $$I_n = \frac{1}{2} x^{n-1} e^{x^2} - \frac{n-1}{2} \int I_{n-2}.$$

   Given that $I_1 = \frac{1}{2} e^{x^2} + C$, use this recursion formula to find I_3 and I_5.

Solution: To prove the formula, use parts with $U(x) = x^{n-1}$ and $dV(x) = xe^{x^2}\,dx$:

$$I_n = U(x)V(x) - \int V(x)dU(x) = \frac{x^{n-1}e^{x^2}}{2} - \frac{n-1}{2}\int x^{n-2}e^{x^2}\,dx \quad \Rightarrow$$

$$I_n = \frac{x^{n-1}e^{x^2}}{2} - \frac{n-1}{2}I_{n-2}.$$

$$I_3 = \frac{1}{2}x^2e^{x^2} - \frac{1}{2}e^{x^2}, \qquad I_5 = \frac{1}{2}x^4e^{x^2} - 2I_3.$$

(e) Evaluate without table $\int e^{3x}\sin\left(\frac{x}{3}\right)\,dx$. (Handout 3)

3. Substitution, trig powers, trig substitution, inverse trig functions, completing the square, recoginizing logarithms ($\int \frac{f'(x)}{f(x)}\,dx$) , natural variable.
Evaluate (Handout 3):

(a)

$$\int \frac{1}{2+x^{\frac{1}{3}}}\,dx.$$

(b)

$$\int_0^{\frac{1}{2}} \frac{2x+1}{4x^2+1}\,dx.$$

(c)

$$\int_0^2 x\sqrt{4x-x^2}\,dx.$$

The following are from various sources:

(d)

$$\int \frac{\ln(x)}{x[1+\ln(x)]}\,dx.$$

Solution: Use substitution with $u = \ln(x), \quad du = \frac{1}{x}\,dx$.

$$= \int \frac{u}{1+u}\,du = \int [1 - \frac{1}{1+u}]\,du = \ln(x)\ \ \ln|1+\ln(x)| + C.$$

(e)

$$\int \tan^2(x)\sec^2(x)\,dx.$$

Solution: Use substitution with $u = \tan(x), \quad du = \sec^2(x)\,dx$:

$$= \int u^2\,du = \frac{1}{3}\tan^3(x) + C.$$

(f)

$$\int x^2 \arcsin(x)\, dx.$$

Solution: Kill the ugly function with parts: $U(x) = \arcsin(x), \quad dV(x) = x^2\, dx$:

$$= \frac{1}{3}x^3 \arcsin(x) - \frac{1}{3}\int \frac{x^3}{\sqrt{1-x^2}}\, dx\,.$$

Now use trig substitution with $x = \sin(u), \quad dx = \cos(u)\, du$:

$$= \frac{1}{3}x^3 \arcsin(x) - \frac{1}{3}\int \sin^3(u)\, du = \frac{1}{3}x^3 \arcsin(x) - \frac{1}{3}\int [1-\cos^2(u)]\sin(u)\, du.$$

For the remaining integral, use substitution with $w = \cos(u), \quad dw = -\sin(u)\, du$ to get

$$\frac{1}{3}x^3 \arcsin(x) + \frac{1}{3}cos(u) + +\frac{1}{9}\cos^3(u) + C.$$

Now draw the triangle defined by $x = \sin(u)$ to get

$$= \frac{1}{3}x^3 \arcsin(x) + \frac{1}{3}\sqrt{1-x^2} + +\frac{1}{9}\left[1-x^2\right]^{\frac{3}{2}} + C.$$

(g)

$$\int \frac{e^x}{1+e^{2x}}\, dx.$$

Solution: Use substitution with $u = e^x, \quad du = e^x\, dx$:

$$= \int \frac{1}{1+u^2}\, du = \arctan\left(e^x\right) + C.$$

(h)

$$\int \frac{e^{2x}}{1+e^x}\, dx.$$

Solution: Use substitution with $u = e^x, \quad du = e^x\, dx$:

$$= \int \frac{u}{1+u}\, du = \int \left[1 - \frac{1}{1+u}\right] du = u - \ln|1+u| + C = e^x + \ln|1+e^x| + C.$$

(i)

$$\int \cos^2(x)\sin^{101}(x)\, dx.$$

Solution: Error! This problem is a bit too hard for final exam drill. Should have been

$$\int \cos^{100}(x)\sin^3(x)\, dx.$$

For this corrected problem, use substitution with $u = \cos(x), \quad du = -\sin(x)\, dx$:

$$= -\int u^{100}[1-u^2]\, du = -\frac{1}{101}\cos^{101}(x) + \frac{1}{103}\cos^{103}(x) + C.$$

(j)

$$\int \frac{1}{[x^2+6x+11]^{\frac{3}{2}}}\,dx.$$

Solution: Complete the square:

$$= \int \frac{1}{[(x+3)^2+2]^{\frac{3}{2}}}\,dx,$$

then use trig substitution $x+3 = \sqrt{2}\tan(u)$ to get

$$= \frac{1}{2}\int \cos(u)\,du = \frac{1}{2}\sin(u) + C.$$

We draw the triangle defined by

$$\frac{x+3}{\sqrt{2}} = \tan(u)$$

to get

$$= \frac{1}{2}\frac{x+3}{\sqrt{(x+3)^2+2}} + C\,.$$

(k)

$$\int \sqrt{1+x-x^2}\,dx.$$

Solution: Complete the square:

$$= \int \sqrt{\frac{3}{4} - (x-\frac{1}{2})^2}\,dx.$$

Now use trig substitution with $x - \frac{1}{2} = \frac{\sqrt{3}}{2}\sin(u)$ to get

$$= \frac{3}{4}\int \cos^2(u)\,du = \frac{3}{8}\int [1+\cos(2u)]\,du = \frac{3}{8}[u + \frac{1}{2}\sin(2u)] + C,$$

where

$$u = \arcsin\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right).$$

This is an acceptable answer.

See also old quizzes and worksheets and the drill problems in the detailed syllabus.

4. Partial Fractions

Evaluate:

(a)

$$\int \frac{x^3+x^2-2}{(x-1)^2)\,(x^2+1)}\,dx.$$

Solution: We use partial fractions:

$$\frac{x^3+x^2-2}{(x-1)^2)\,(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}.$$

Putting the right side over a common denominator and equating the resulting numerator to the left side numerator, we get

$$x^3+x^2-2 = A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2.$$

Setting $x = 1$ we see that $0 = 2B$, so $B = 0$, which we insert into the above equation to get

$$x^3+x^2-2 = A(x-1)(x^2+1) + (Cx+D)(x-1)^2.$$

Now equate the coefficients of (say) x^3, $\quad x^2$, and the constant terms to get three equations for the remaining unknowns:

$$1 = A + C, \quad 1 = -A - 2C + D, \quad -2 = -A + D.$$

Solving, we get:

$$C = -\frac{3}{2}, \quad A = \frac{5}{2}, \quad D = \frac{1}{2}.$$

Thus our integral becomes

$$\int \left[\frac{A}{x-1} + \frac{Cx+D}{x^2+1}\right] dx = \frac{5}{2}\ln|x-1| - \frac{3}{4}\ln\left(x^2+1\right)\frac{1}{2}\arctan(x) + C.$$

(b)

$$\int \frac{3x-1}{x(x-1)(x+1)}\,dx.$$

Solution: We use partial fractions:

$$\frac{3x-1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} \quad \Rightarrow$$

$$3x-1 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1),$$

and we can easily solve by substitution successively: $x = 0, -1, 1$:

$$A = 1, \quad C = 2, \quad B = 1,$$

and our integral becomes

$$\int \left[\frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}\right] dx =$$

$$= \ln|x| + \ln|x-1| + 2\ln|x+1| + C = \ln\left[|x(x-1)(x+1)^2|\right] + C.$$

5. Improper Integrals

Explain why the integral is improper, and decide whether convergent or divergent:

(a)

$$\int_0^{\frac{\pi}{2}} \sec^2(x)\,dx.$$

Solution: Improper because the integrand is singular (has a vertical asymptote) at $x = \frac{\pi}{2}$. We try to evaluate

$$= \lim_{T \to \frac{\pi}{2}-} \int_0^T \sec^2(x)\,dx = \lim_{T \to \frac{\pi}{2}-} \left[\tan(x)\,|_0^T\right] = \lim_{T \to \frac{\pi}{2}-} \left[\tan(T)\,\right] = +\infty.$$

The integral does not converge.

(b)

$$\int_0^{\infty} \sin(x)\,dx.$$

Solution: The integral is improper because we are trying to integrate over an unbounded interval.

$$= \lim_{T \to \infty} \int_0^T \sin(x)\,dx = \lim_{T \to \infty} \left[-\cos(T) + 1\right],$$

and this limit does not exist because $\cos(T)$ oscillates between -1 and $+1$ as $T \to \infty$.

(c)

$$\int_{-1}^{1} \frac{x}{\sqrt{x+1}}\,dx.$$

Solution: Improper because the integrand has a vertical asymptote at $x = -1$.

$$= \lim_{a \to -1+} \int_a^1 \frac{x}{\sqrt{x+1}}\,dx.$$

We use substitution with $x + 1 = u^2, \quad dx = 2u\,du$:

$$= \lim_{a \to -1+} \int_{\sqrt{a+1}}^{\sqrt{2}} 2\frac{u(u^2-1}{u}\,du = 2 \lim_{a \to -1+} \left\{ \left[\frac{1}{3}(x+1)^{\frac{3}{2}} - \sqrt{x+1} \right] \,|_a^1 \right\} =$$

$$= 2 \lim_{a \to -1+} \left\{ \frac{1}{3}2^{\frac{3}{2}} - \sqrt{2} + \frac{1}{3}(a+1)^{\frac{3}{2}} - \sqrt{a+1} \right\} = \frac{1}{3}2^{\frac{5}{2}} - 2\sqrt{2}.$$

The integral converges.

(d)

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx.$$

Solution: The integral is improper because we are trying to integrate over an unbounded interval.

$$= \lim_{T\to\infty} \int_1^T \frac{1}{1+x^2}\,dx = \lim_{T\to\infty} \left[\arctan(x)\,|_1^T\right] = \lim_{T\to\infty} \left[\arctan(T) - \frac{\pi}{4}\right] = \frac{\pi}{4}.$$

The integral converges.

6. Volume by discs and shells. Consider the region bounded by $y = x + \ln(x)$ and the x-axis, $1 \le x \le e$. Find the volume of the solid obtained by

   (a) rotating this region about the x-axis, using discs;
   Solution:
   Correction: In the originally posted solutions, I use $f(x) = \ln(x)$ rather than $f(x) = x + \ln(x)$, in all of the solutions in this section. In this corrected posting, the $\ln(x)$ has been replaced by the correct $[x + \ln(x)]$ in all of the solutions, but I did not calculate all of the resulting integrals, just sketched how to evaluate the more difficult ones.

   $$V = \pi \int_1^e [x + \ln(x)]^2\,dx.$$

   To evaluate, expand the square and use parts, for example the last integral will be $\int \ln^2(x)\,dx$ so use parts with $U(x) = \ln^2(x), \quad dV(x) = 1\,dx$:

   $$= \pi x \ln^2(x)\,|_1^e - 2\pi \int_1^e \ln(x)\,dx.$$

   Now use parts again with $U(x) = \ln(x), \quad dV(x) = 1\,dx$

   $$= \pi e - 2\pi[x\ln(x) - x]\,|_1^e = \pi e - 2\pi[e - e + 1] = \pi[e - 2].$$

   For the integral of $2x\ln(x)$ use parts in the same way.

   (b) rotating this region about the y-axis, using shells;
   Solution:

   $$V = 2\pi \int_1^e x[x + \ln(x)]\,dx.$$

   We integrate the one integral of $x\ln(x)$ by parts, with $U(x) = \ln(x), \quad dV(x) = x\,dx$ to get (for this one integral, the other one is easy):

   $$= 2\pi[\frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2]\,|_1^e = 2\pi[\frac{1}{4}e^2 + \frac{1}{4}].$$

   (c) rotating this region about the line $y = 9$, using discs;
   Solution (set-up only):

   $$V = \pi \int_1 e \left[9^2 - (9 - x - \ln(x))^2\right]\,dx$$

(d) rotating this region about the line $x = 3$, using shells.
Solution (set-up only):

$$V = 2\pi \int_1^e (3 - x)[x + \ln(x)]\, dx.$$

7. Arclength and surface area.

(a) Write down the integral for the arclength of the curve defined by $x^2 y + x = 1$.
Solution: Forgot to give an interval. Use $1 \le x \le 3$. We have

$$f(x) = \frac{1-x}{x^2}, \qquad f'(x) = -2x^{-3} + x^{-1} = \frac{x^2 - 2}{x^3}.$$

Then the arclength is

$$s = \int_1^3 \left[1 + \left(\frac{x^2-2}{x^3}\right)^2\right]^{\frac{1}{2}} dx.$$

(b) Write down the integral for the surface area obtained by rotating the curve from the previous problem about the y-axis.
Solution:

$$\mathcal{A} = 2\pi \int_1^3 x \left[1 + \left(\frac{x^2-2}{x^3}\right)^2\right]^{\frac{1}{2}} dx.$$

(c) Same as above, but rotate about the line x = -2.
Solution:

$$\mathcal{A} = 2\pi \int_1^3 (x+2) \left[1 + \left(\frac{x^2-2}{x^3}\right)^2\right]^{\frac{1}{2}} dx\ .$$

(d) Same as above, but rotate about the x-axis.
Solution:

$$\mathcal{A} = 2\pi \int_1^3 \frac{1-x}{x^2} \left[1 + \left(\frac{x^2-2}{x^3}\right)^2\right]^{\frac{1}{2}} dx.$$

8. Taylor Polynomials (see Handout 5)

(a) Find the Taylor Polynomial $P_3(x)$ about $a = \frac{\pi}{2}$ for $f(x) = \ln(\sin(x))$.
Solution: We have

$$f(x) = \ln(\sin(x)), \quad f'(x) = \cot(x), \quad f''(x) = -\csc^2(x), \quad f^{(3)}(x) = 2\csc^2(x)\cot(x),$$

and we evaluate at $\frac{\pi}{2}$:

$$f\left(\frac{\pi}{2}\right) = 0, \quad f'\left(\frac{\pi}{2}\right) = 0, \quad f''\left(\frac{\pi}{2}\right) = -1, \quad f^{(3)}\left(\frac{\pi}{2}\right) = 0.$$

Then

$$P_3(x) = -\frac{1}{2}\left(x - \frac{\pi}{2}\right)^2.$$

(b) Derive the Taylor Polynomial of any degree for $f(x) = \sin(x)$ about $x = 0$.
Solution: See your text.

9. Ordinary Differential Equations
Solve:

(a) $y'(x) + y(x)\sin x = \cos(x)\sin(x), \qquad y(2) = 3$. (Handout 5) Solution: See the handout.

(b) Find the general solution, in explicit form, of $(x+1)y' + (1+y)x^2 = 0$. (Handout 5)

(c) Find the solution, in the simplest form, of the initial value problem $y' = \frac{y}{x-y}, \quad y(2) = 1$. (Handout 5)

(d) Find the general solution of $y'(x) - 2xy(x) = x$. (Handout 5)

(e) $(x^2+1)\,y' + y = 0$.
Solution: Both linear and separable, can solve either way. Linear:

$$y' + \frac{1}{(x^2+1)}y = 0,$$

so the normalized equation has the coefficient of $y(x)$ as

$$a(x) = \frac{1}{(x^2+1)} \quad \Rightarrow \quad A(x) = \int a(x)\,dx = \arctan(x),$$

and so the integrating factor is

$$e^{A(x)} = e^{\arctan(x)},$$

and we multiply the nomalized equation by this I.F.:

$$e^{\arctan(x)}[y' + \frac{1}{(x^2+1)}y] = 0 \qquad \Rightarrow \left\{e^{\arctan(x)}y(x)\right\}' = 0,$$

so

$$e^{\arctan(x)}y(x) = C \qquad \Rightarrow y(x) = Ce^{-\arctan(x)}.$$

Separable:

$$\frac{y'(x)}{y(x)} = -\frac{1}{(x^2+1)} \qquad \Rightarrow$$

$$\ln(y(x)) = -\arctan(x) + C \quad \Rightarrow \quad y(x) = e^C e^{-\arctan(x)}.$$

(f) $y' - 3y = e^{2x}, \qquad y(0) = 4$.
Solution: Linear. Already normalized, with $a(x) = -3$ so

$$A(x) = -3x \quad e^{A(x)} = e^{-3x}.$$

So we multiply by the integrating factor to get

$$e^{-3x}\,[y' - 3y] = e^{-x} \qquad \Rightarrow \qquad \left\{e^{-3x}y(x)\right\}' = e^{-x} \quad \Rightarrow$$

$$e^{-3x}y(x) = -e^{-x} + C. \qquad y(x) = -e^{2x} + Ce^{3x}.$$

Setting $x = 0$ we have $y(0) = 4 = -1 + C$, so $C = 5$.

(g) $y''(x) - 5y'(x) + 6y(x) = 0.$
Solution: Second order linear with constant coefficients. Guess $y(x) = e^{mx}$ to get the characteristic equation:

$$m^2 - 5m + 6 = 0 \quad \Rightarrow \quad (m-2)(m-3) = 0 \quad \Rightarrow \quad m = 2, 3.$$

So the general solution is

$$y(x) = C_1 e^{2x} + C_2 e^{3x}.$$

(h) $y''(x) - 5y'(x) + 6y(x) = 3e^{4x}.$
Solution: We have the general solution of the homogeneous equation (right side zero) from the previous problem. Therefore we need only find one particular solution, $y_p(x)$, which solves the given non-homogeneous equation. We guess $y_p(x) = Ae^{4x}$ and plug this into the equation (factoring out the term e^{4x} which occurs in every term on the left:

$$[16A - 20A + 6A]e^{4x} = 3e^{4x}.$$

Thus $A = \frac{3}{2}$, and the general solution of our equation is

$$y(x) = \frac{3}{2}e^{4x} + C_1 e^{2x} + C_2 e^{3x}.$$

(i) $y''(x) - 5y'(x) + 6y(x) = 2x^2 + 3x - 5.$
Solution: Same homogeneous equation as before, only the forcing term (non-homogeneous term) has changed. So now we have to guess $y_p(x) = Ax^2 + Bx + C$. Plugging this into the equation we get:

$$2A - 5\,[2Ax + B] + 6\left[Ax^2 + Bx + C\right] = 2x^2 + 3x - 5$$

and we can quickly solve for A, B, and C by equating coefficients of, respectively, x^2, x and the constant term:

$$A = \frac{1}{3}, \qquad B = \frac{19}{18}, \qquad C = -\frac{7}{108}.$$

Thus the general solution is

$$y(x) = \frac{1}{3}x^2 + \frac{19}{18}x + -\frac{7}{108} + C_1 e^{2x} + C_2 e^{3x}.$$

(j) $y''(x) - 4y'(x) + 4y(x) = 0.$
Solution: The characteristic equation is

$$m^2 - 4 = 0 \qquad m = 2,\ -2.$$

Thus the general solution of this homogeneous equation is

$$y(x) = C_1 e^{2x} + C_2 e^{-2x}.$$

(k) $2y''(x) - 3y'(x) + 2y(x) = 0$

Solution: The characteristic equation is

$$2m^2 - 3m + 2 = 0 \quad \Rightarrow \quad m = \frac{3 \pm \sqrt{9-16}}{4} \quad \Rightarrow \quad m = \frac{3}{4} \pm \iota \frac{\sqrt{5}}{4}.$$

Then the general solution is

$$y(x) = C_1 e^{\frac{3}{4}x} \cos\left(\frac{\sqrt{5}}{4}x\right) + C_2 e^{\frac{3}{4}x} \sin\left(\frac{\sqrt{5}}{4}x\right) \quad .$$

(l) $y''(x) - y(x) = \cos(3x)$.

Solution: The characteristic equation for the homogeneous equation is $m^2 - 1 = 0$ so the roots are $m = -1$ and $m = +1$. Thus the general solution of the homogeneous equation is $y(x) = C_1 e^{-x} + C_2 e^x$. We guess a particular solution $y_p(x) = A\cos(3x) + B\sin(3x)$ to the nonhomogeneous equation. Plugging this into the equation, we get

$$-9A\cos(3x) - 9B\sin(3x) - A\cos(3x) - B\sin(3x) = \cos(3x).$$

Thus we get two equations by matching the coefficients of $\cos(3x)$ and $\sin(3x)$, respectively:

$$-9A - A = 1, \qquad -9B - B = 0 \qquad \Rightarrow \qquad A = \frac{1}{10}, \quad B = 0.$$

So the general solution of our equation is

$$y(x) = \frac{1}{10}\cos(3x) + C_1 e^{-x} + C_2 e^x \quad .$$