MATH 253 Handout #1 Solution.

For 1)

For $x > 0$ $\int \sqrt{x}\left(\frac{5}{\sqrt{x}} - \frac{4}{x^{\frac{3}{2}}}\right)dx = 5\int \frac{\sqrt{x}}{\sqrt{x}}dx - 4\int \frac{\sqrt{x}}{x^{\frac{3}{2}}}dx =$

$= 5\int dx - 4\int x^{-1}dx = 5x - 4\ln x + c, x > 0$

For 2)

use substitution $u = \sin x, du = \cos x\ dx, \cos^2 x = 1 - \sin^2 x$

and if $x = \frac{\pi}{4}$, then $u = \frac{1}{\sqrt{2}}$,and if $x = \frac{\pi}{2}, u = 1$, so

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos^3 x}{\sin^3 x}dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos^2 x}{\sin^3 x} \cdot \cos x dx = \int_{\frac{1}{\sqrt{2}}}^{1} \frac{(1-u^2)}{u^3}du = \int_{\frac{1}{\sqrt{2}}}^{1} \frac{1}{u^3}du - \int_{\frac{1}{\sqrt{2}}}^{1} \frac{1}{u}du =$$

$$= \left[\frac{u^{-2}}{-2}\right]_{\frac{1}{\sqrt{2}}}^{1} - [\ln|u|]_{\frac{1}{\sqrt{2}}}^{1} = \frac{-1}{2}[1-2] - \left[\ln 1 - \ln 2^{-\frac{1}{2}}\right] = \frac{1}{2} - \left[0 + \frac{1}{2}\ln 2\right] = \frac{1}{2} - \frac{1}{2}\ln 2.$$

For 3)

$D_f = (-\infty, 4]$ since we must have $(4-x) \geq 0$ and $R_f \subset (-\infty, 0)$
since the exponential function has always positive values.
now to find the inverse solve for x: $y = e^{\sqrt{4-x}}$,
if y is positive we can apply ln to both sides
$\ln y = \sqrt{4-x}$ and again the right-hand side is positive or 0
so **IF** $\ln y \geq 0$ we can square both sides $(\ln y)^2 = 4 - x$ and $x = 4 - \ln^2 y$,
finally by interchanging x and y
$y = f^{-1}(x) = 4 - \ln^2 x, D_{f^{-1}} = R_f =?$,and $R_{f^{-1}} = D_f = (-\infty, 4]$
to get the range of f we have to solve when $\ln y \geq 0$ and that is for $y \geq 1$
from the graph of ln
OR by applying exp. function to the inequality and using $e^0 = 1$
thus $R_f = D_{f^{-1}} = [1, +\infty)$.

For 4)

For $x \neq 0$ $\int \frac{5x - \sqrt[3]{x} + 3}{\sqrt[3]{x}}dx = \int \left(\frac{5x}{\sqrt[3]{x}} - 1 + \frac{3}{x^{\frac{1}{3}}}\right)dx = 5\int x^{\frac{2}{3}}dx - \int dx + 3\int x^{-\frac{1}{3}}dx =$

$= 5 \cdot \frac{3}{5}x^{\frac{5}{3}} - x + 3 \cdot \frac{3}{2}x^{\frac{2}{3}} + c = 3x^{\frac{5}{3}} - x + \frac{9}{2}x^{\frac{2}{3}} + c.$

For 5)

use substitution $u = 3 - x, du = -\ dx, x = 3 - u$ and $x^2 = (3-u)^2$

if $x = 0$, then $u = 3$, and if $x = 2, u = 1$, so

$$\int_0^2 \frac{x^2}{3-x}dx. = -\int_3^1 \frac{(3-u)^2}{u}du = \int_1^3 \frac{9 - 6u + u^2}{u}du = \int_1^3 \left(\frac{9}{u} - 6 + u\right)du =$$

$$= 9[\ln u]_1^3 - 6[u]_1^3 + \frac{1}{2}[u^2]_1^3 = 9[\ln 3 - \ln 1] - 6[3-1] + \frac{1}{2}[9-1] = 9\ln 3 - 12 + 4 = 9\ln 3 - 8.$$

For 6)

$D_f = (-\infty, 1)$ since we must have $\frac{1}{1-x} > 0$ so $1 - x > 0$ thus $1 > x$
and $R_f = (-\infty, +\infty)$ since logarithmic function has positive and negative values.
now to find the inverse solve for x: $y = \ln\frac{1}{1-x} = -\ln(1-x)$,

$-y = \ln(1-x)$ apply exp. function to both sides -possible for any y

(OR $e^y = \frac{1}{1-x}, \frac{1}{e^y} = 1-x, e^{-y} = \frac{1}{e^y}$)$e^{-y} = 1-x$ and $x = 1-e^{-y}$
finally by interchanging x and y
$y = f^{-1}(x) = 1-e^{-x}, D_{f^{-1}} = R_f = (-\infty, +\infty)$,and $R_{f^{-1}} = D_f = (-\infty, 1)$.

For 7)

using $\quad (A-B)^2 = A^2 - 2AB + B^2$

$\int \left(2\sqrt{x} - \frac{1}{x}\right)^2 dx = \int 4x\ dx - 4\int \frac{\sqrt{x}}{x}dx + \int x^{-2}dx = 4 \cdot \frac{x^2}{2} - 4\int x^{-\frac{1}{2}}dx - x^{-1} + c =$

$= 2x^2 - 8x^{\frac{1}{2}} - \frac{1}{x} + c$ for $x > 0$

For 8)

use substitution $u = \frac{1}{x}, du = -x^{-2}dx$,so $\quad -du = \dfrac{dx}{x^2}$ and if $x = \frac{1}{2}$, then $u = 2$,
and if $x = 1, u = 1$ so

$$\int_{\frac{1}{2}}^{1} \frac{3^{\frac{1}{x}}}{x^2}dx = -\int_{2}^{1} 3^u du = \left[\frac{3^u}{\ln 3}\right]_1^2 = \frac{1}{\ln 3}\left[9-3\right] = \frac{6}{\ln 3}.\text{(using } -\int_b^a = \int_a^b \text{)}$$

For 9)

$D_f = \{x \neq -3\}$ but $R_f =?$, to find the inverse solve for x: $y = \dfrac{1-2x}{x+3}$

for $x \neq -3$ $\;y\,(x+3) = 1-2x$,so $xy + 3y = 1-2x$, and $\quad xy + 2x = 1-3y$,

so $\quad x\,(y+2) = 1-3y$.To be able to solve for x we have to assume that $(y+2) \neq 0$,

so $y \neq -2$.Therefore $R_f = \{y \neq -2\}$ and we can finish solving : $x = \dfrac{1-3y}{y+2}$

$(\ x \longleftrightarrow y)\quad f^{-1}(x) = \dfrac{1-3x}{x+2}, D_{f^{-1}} = R_f = \{x \neq -2\}$,and $R_{f^{-1}} = D_f = \{y \neq -3\}$.

For 10)

$\int x^{\frac{1}{3}}\,(2-x)\,dx = 2\int x^{\frac{1}{3}}dx - \int x \cdot x^{\frac{1}{3}}dx = 2 \cdot \frac{3}{4}x^{\frac{4}{3}} - \int x^{\frac{4}{3}}dx = \frac{3}{2}x^{\frac{4}{3}} - \frac{3}{7}x^{\frac{7}{3}} + c$

For 11)

use substitution $u = x^2, du = 2xdx$, and if $x = -1$, then $u = 1$,
and if $x = 0, u = 0$,so

$$\int_{-1}^{0} xe^{-x^2}dx = \frac{1}{2}\int_1^0 e^{-u}du = \frac{1}{2}\left[\frac{e^{-u}}{-1}\right]_1^0 = -\frac{1}{2}\left(1-e^{-1}\right) = \frac{1}{2}\left(\frac{1}{e} - 1\right)$$

For 12)

$D_f = [-1, +\infty)$since we must have $(1+x) \geq 0$ and $R_f = (-\infty, 0]$
now to find the inverse solve for x: $y = -\sqrt{1+x}$,
$-y = \sqrt{1+x}$, so $-y$ must be positive or 0
$y \leq 0$ then we can square both sides and $y^2 = 1+x$, and $x = y^2 - 1$
so $f^{-1}(x) = x^2 - 1, D_{f^{-1}} = R_f = (-\infty, 0]$,and $R_{f^{-1}} = D_f = [-1, +\infty)$