Solutions to Lab Quiz 4, Math 253 B31/B32 Friday 04/01 , 1300 hours

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1. Explain why the integrals are improper, and decided (with justification) whether convergent or divergent. If convergent, evaluate:

$$\text{(a)} \int_0^{\frac{\pi}{2}} \frac{\cos(x)}{1-\sin(x)}\,dx \quad \text{(b)} \int_2^{\infty} \frac{1}{1+x^2}\,dx.$$

Solution: (a) The integral is improper because the integrand has a vertical asymptote at $x = \frac{\pi}{2}$. We attempt to calculate it:

$$= \lim_{T\to\frac{\pi}{2}-} \int_0^T \frac{\cos(x)}{1-\sin(x)}\,dx = \lim_{T\to\frac{\pi}{2}-} \ln|1-\sin(x)|\,|_0^T = \lim_{T\to\frac{\pi}{2}-} \ln|1-\sin(T)| = -\infty.$$

The integral diverges.

(b) The integral is improper because the interval of integration is infinite. We attempt to evaluate:

$$= \lim_{T\to\infty} ` \int_2^T \frac{1}{1+x^2}\,dx = \lim_{T\to\infty} \arctan(T) - \arctan(2) = \frac{\pi}{2} - \arctan(2).$$

So the integral converges.

2. Use comparison to decide whether the following integral is convergent or divergent:

$$\int_1^{\infty} \frac{2+\cos(x)}{x^{\frac{3}{2}}}\,dx.$$

Solution: We suspect that the integral converges since the numerator oscillates between 1 and 3 and the denominator involves x^p with $p > 1$. So we need to bound the integrand above by something that converges. We have

$$\cos(x) \le 1 \quad \Rightarrow \quad 2+\cos(x) \le 3 \quad \Rightarrow \quad \frac{2+\cos(x)}{x^{\frac{3}{2}}} \le \frac{3}{x^{\frac{3}{2}}}.$$

Since $\int_1^{\infty} \frac{3}{x^{\frac{3}{2}}}\,dx$ converges, so does our integral, by comparison.

3. Use the trapezoidal rule with n = 3 to approximate the integral

$$\int_1^2 \frac{x}{1+x^2}\,dx.$$

Evaluate the integral exactly and compare your results.

Solution:

$$T_3 = \frac{2-1}{3}\left[\frac{1}{2}f(1) + f\left(\frac{4}{3}\right) + f\left(\frac{5}{3}\right) + \frac{1}{2}f(2)\right] =$$

$$= \frac{1}{3}\left[\frac{1}{2}\cdot\frac{1}{2} + \frac{12}{25} + \frac{15}{34} + \frac{1}{2}\cdot\frac{2}{5}\right] = 0.457$$

The exact value is

$$\frac{1}{2}\ln(1+x^2)\,|_1^2 = \frac{1}{2}\ln(5/2) = .458\ ,$$

so the trapezoidal approximation is low by about 0.001.