Lab Worksheet 2 Solutions, Math 253 B25-32, February7-11, 2005

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1. Find the area inside both of the circles $x^2 + y^2 = 1$ and $(x-1)^2 + y^2 = 1$.
   Note: Sketch them first. They intersect at $(\frac{1}{2}, \pm\frac{\sqrt{3}}{x})$. Integrate on the y-axis, and use trig substitution to evaluate.

   Solution: From a simple sketch of the two circles, we see that for a given value of y, the corresponding x values on the circles are respectively:

   $$x = \pm\sqrt{1-y^2}, \quad x = 1 \pm \sqrt{1-y^2}.$$

   The area we seek is given by

   $$\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \left[\sqrt{1-y^2} - \left(1 - \sqrt{1-y^2}\right)\right] dy = \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} 2\sqrt{1-y^2}\, dy \,\Big|_{\substack{y=\sin(u)\\ dy=\cos(u)\,du}} - y\,\Big|_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} =$$

   $$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 2\cos^2(u)\,du - \sqrt{3} = \left[u + \frac{\sin(2u)}{2}\right] \Big|_{-\frac{\pi}{3}}^{\frac{\pi}{3}} - \sqrt{3} = \frac{2\pi}{3} + 2\sin\left(\frac{2\pi}{3}\right) - \sqrt{3} =$$

   $$= \frac{\pi}{3}.$$

2. Let $I_p = \int \ln^p(x)\,dx$. Use integration by parts to derive the recursion formula $I_p = x\ln^p(x) - pI_{p-1}$. Use this formula to evaluate $\int_1^{e^2} \ln^3(x)\,dx$.
   Solution: Use integration by parts with

   $$U = \ln^p(x),\ dV = 1,\ dU = \frac{p}{x}\ln^{p-1}(x),\ V = x,$$

   the rest is straightforward.

3. Evaluate the following integrals:

   $$(a) \int \frac{1}{x^{\frac{1}{2}} - x^{\frac{1}{3}}}\,dx, \quad (b) \int \frac{x+1}{[(x-2)^2+1]^{\frac{3}{2}}}\,dx,$$

   $$(c) \int \frac{x^2+2}{x\sqrt{x^2-1}}\,dx \quad (d) \int x\arcsin(x)\,dx.$$

   Notes: All answers should be stated in terms of the original variable of integration.
   (a) think natural variable to clear both radicals. You might find it useful to use

   $$\frac{u^3}{u-1} = u^2 + u + 1 + \frac{1}{u-1}.$$

   (b) think $1 + \tan^2(\theta) = \sec^2(\theta)$. (c) sec and ye shall find.
   (d) kill the ugly function.

Solution: (a) Use $x = u^6$, $dx = 6u^5\,du$. You get

$$\int \frac{1}{u^3 - u^2}\,6u^5\,du = \int \frac{u^3}{u-1}\,du =$$

$$= \int \{u^2 + u + 1 + \frac{1}{u-1}\}\,du = \frac{1}{3}u^3 + \frac{1}{2}u^2 + u + \ln|\,u-1\,| + C.$$

$$= \frac{1}{3}\sqrt{x} + \frac{1}{2}x^{\frac{1}{3}} + x^{\frac{1}{6}} + \ln|x^{\frac{1}{6}} - 1| + C.$$

(b) Use $x - 2 = \tan(u)$, $dx = \sec^2(u)\,du$. You get

$$\int \frac{\tan(u) + 3}{[\tan^2(u) + 1]^{\frac{3}{2}}}\,\sec^2(u)\,du = \int [\sin(u) + 3\cos(u)]\,du =$$

$$= -\cos(u) + 3\sin(u) + C = \frac{3(x-2)+1}{\sqrt{(x-2)^2+1}} + C.$$

(c) Use $x = \sec(u)$, $dx = \sec(u)\tan(u)\,du$. You get

$$\int \frac{\sec^2(u) + 2}{\sec(u)\sqrt{\sec^2(u) - 1}}\,\sec(u)\tan(u)\,du = \int [\sec^2(u) + 2]\,du =$$

$$= \tan(u) + 2u + C = \sqrt{x^2 - 1} + 2\arcsin(\sqrt{x^2-1}) + C.$$

(d) Use parts, $U = \arcsin(x), dV = x$, then can do the resulting integral :

$$= \frac{1}{2}x^2\arcsin(x) - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}\,dx|_{\substack{x=\sin(u)\\ dx=\cos(u)du}} = \frac{1}{2}x^2\arcsin(x) - \frac{1}{2}\int \frac{\sin^2(u)}{\cos(u)}\,\cos(u)\,du =$$

$$= \frac{1}{2}x^2\arcsin(x) - \frac{1}{2}\int \frac{1-\cos(2u)}{2\cos(u)}\,\cos(u)\,du = \frac{1}{2}x^2\arcsin(x) - \frac{1}{4}[u - \frac{1}{2}\sin(2u)] + C =$$

$$= \frac{1}{2}x^2\arcsin(x) - \frac{1}{4}\arcsin(x) + \frac{1}{8}2x\sqrt{1-x^2} + C.$$