Mathematics 253 Lab Worksheet, Labs B25-B32

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1. Consider the function $f(x) = e^{\sin(x)}$.
   (a) Find the Taylor Polynomial $P_3(x;\, 0)$ for $f(x)$ based at $x = 0$.
   (b) Find the Taylor Polynomial $P_3\left(x;\, \frac{\pi}{2}\right)$ for $f(x)$ based at $x = \frac{\pi}{2}$.
   (c) Use the Taylor Polynomials found in (a) and (b) to estimate $f\left(\frac{\pi}{4}\right)$, and compare with the exact value.

2. Find the general solution of the following differential equations:

$$\text{(a)}\quad (1 + e^x)\, y'(x) = \cos^2(y(x))\, e^x,$$
$$\text{(b)}\quad xy'(x) - y(x) + x^2 \ln(x) = 0.$$

3. (a) Find the general solution of

$$y''(x) - y'(x) - 2y(x) = 0.$$

   (b) Find the solution of the initial value problem

$$y''(x) - y'(x) - 2y(x) = 4x^2 - 4x, \qquad y((0) = 1, \quad y'(0) = 2.$$

Solutions:
1(a)

$$f(x) = e^{\sin(x)}, \quad f'(x) = \cos(x)e^{\sin(x)}, \quad f''(x) = \left(-\sin(x) + \cos^2(x)\right) e^{\sin(x)},$$
$$f^{(3)}(x) = \left(-\cos(x) - 3\sin(x)\cos(x) + \cos^3(x)\right) e^{\sin(x)}.$$

Evaluating at $x = 0$, we get:

$$f(0) = 1, \quad f'(0) = 1, \quad , f^{('')}(0) = 1, \quad f^{(3)}(0) = 0,$$

so the Taylor Polynomial for $f(x)$ about 0 is:

$$P_3(x; 0) = 1 + x + \frac{1}{2}x^2.$$

(b) We evaluate the derivatives at $x = \frac{\pi}{2}$:

$$f\left(\frac{\pi}{2}\right) = e, \quad f'\left(\frac{\pi}{2}\right) = 0, \quad f''\left(\frac{\pi}{2}\right) = -e, \quad f^{(3)}\left(\frac{\pi}{2}\right) = 0,$$

so the Taylor Polynomial for $f(x)$ about $x = \frac{\pi}{2}$ is

$$P_3\left(x; \frac{\pi}{2}\right) = e + \frac{-e}{2!}\left(x - \frac{\pi}{2}\right)^2.$$

(c) Using a calculator or Maple, we find that:

$$P_3(x; 0)\,|_{x=\frac{\pi}{4}} = 2.093823302, \qquad P_3\left(x; \frac{\pi}{2}\right) = 1.879895381, \qquad f\left(\frac{\pi}{4}\right) = 2.028114981.$$

So the polynomial based at $\frac{\pi}{2}$ doesn't do very well.

2(a) The equation is separable, and can be written

$$\sec^2(y(x))\,y'(x) = \frac{e^x}{e^x+1}.$$

To integrate the left side substitute $u = y(x)$, to integrate the right side substitute $u = e^x$, when the dust clears you have

$$\tan(y(x)) = \ln(e^x+1) + C, \qquad \text{or} \qquad y(x) = \arctan(\ln(e^x+1)+C).$$

(b) This equation is linear, but not normalized. The normalized form is

$$y'(x) - \frac{1}{x}y(x) = -x\ln(x).$$

We compute the integrating factor, which is the exponential of the antiderivative of the coefficient of $y(x)$:

$$I.F. = \exp\left(\int [\frac{-1}{x}]\,dx\right) = \exp(-\ln(x)) = \frac{1}{x}.$$

Multiplying the NORMALIZED equation by the I.F., we get

$$\frac{1}{x}y'(x) - \frac{1}{x^2}y(x) = -\ln(x), \qquad \text{equivalently} \qquad \left(\frac{1}{x}y(x)\right)' = -\ln(x)\,dx.$$

The right side can be integrated using integration by parts, with
$U(x) = \ln(x),\ \ dV(x) = 1\,dx$:

$$\frac{1}{x}y(x) = -x\ln(x) + x + C \quad \Rightarrow y(x) = -x^2\ln(x) + x^2 + Cx.$$

3. (a)

$$y''(x) - y'(x) - 2y(x) = 4x^2 - 4x, \qquad y((0) = 1, \quad y'(0) = 2.$$

This equation is second order linear, so we guess a solution of the form $y(x) = e^{mx}$ which gives us the characteristic equation:

$$m^2 - m - 2 = 0 \quad \Rightarrow \quad m = 2, \quad m = -1.$$

Then the general solution is $y(x) = C_1e^{-x} + C_2e^{2x}$.
(b) We know that the general solution of this non-homogeneous equation can be written $y(x) = C_1e^{-x} + C_2e^{2x} + y_p(x)$, where $y_p(x)$ is any particular solution of this equation. We also know that when the right side is a polynomial of degree 2, we should make the guess $y_p(x) = Ax^2 + Bx + C$ and if we plug this function into the equation we get

$$2A - [2Ax+B] - 2[Ax^2+Bx+C] = 4x^2 - 4x \quad \Rightarrow \quad -2A = 4 \ \ ((\text{coefficients of } x^2),$$

$$-2A - 2B = -4 \ \ (\text{coefficients of } x), \quad 2A - B - C = 0, \ \ (\text{constant terms}).$$

Solving, we have $A = -2,\ B = 4,\ C = -4$, so the general solution is

$$y(x) = C_1e^{-x} + C_2e^{2x} - 2x^2 + 4x - 4.$$

Setting $y(0) = 1, \; y'(0) = 2$, we get two equations for C_1 and C_2:

$$C_1 + C_2 - 4 = 1, \qquad -C_1 + 2C_2 + 4 = 2 \quad \Rightarrow \quad C_1 = 4, \quad C_2 = 1,$$

so the solution of the ivp is

$$y(x) = 4e^{-x} + e^{2x} - 2x^2 + 4x - 4.$$