Examples of Second Order Linear Differential Equations

First of all, there are two different “species” of such equations, the homogeneous ones (where the right-hand side is 0) and the non-homogeneous ones (where the right-hand side is different from 0).

Homogeneous Equations

The basic idea is to find two *different* solutions $y_1(x)$ and $y_2(x)$ and then the general solution is

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$

for *some* constants C_1 and C_2.

You do this by “hunting” for an exponential solution of the form

$$y(x) = e^{\lambda x}.$$

For example, to solve $y'' - 6y' + 8y = 0$, substitute in $y = e^{\lambda x}$ to obtain

$$\begin{aligned} y'' - 6y' + 8y &= 0 \\ \lambda^2 e^{\lambda x} - 6\lambda e^{\lambda x} + 8e^{\lambda x} &= 0 \\ e^{\lambda x}\{\lambda^2 - 6\lambda + 8\} &= 0 \\ \lambda^2 - 6\lambda + 8 &= 0. \end{aligned}$$

Solving for λ, we have

$$(\lambda - 2)(\lambda - 4) = 0$$

so that, $\lambda = 2, 4$. This means that

$$y_1(x) = e^{2x} \quad \text{and} \quad y_2(x) = e^{4x}$$

are two different solutions and so the general solution is

$$y(x) = C_1 e^{2x} + C_2 e^{4x},$$

for *some* constants C_1 and C_2.

We can take a short cut here – notice that the coefficients of the quadratic in λ,

$$\lambda^2 - 6\lambda + 8$$

are *exactly* the same as the corresponding coefficients of the original differential equation

$$y'' - 6y' + 8y = 0.$$

This will always happen and so you can just skip the substituting $y = e^{\lambda x}$ step and go directly to the right quadratic in λ. By the way, this quadratic has a name – it's called the *characteristic* equation. Here is a summary of what can happen.

- **Distinct Roots.** The example we just finished is of this type. Here is another. Let's solve the equation $y'' + 8y' + 15 = 0$. The characteristic equation is

$$\lambda^2 + 8\lambda + 15 = 0$$

which we factor to obtain

$$(\lambda+3)(\lambda+5)=0$$

and so $\lambda=-3,-5$. Hence,

$$y_1(x)=e^{-3x} \quad \text{and} \quad y_2(x)=e^{-5x}$$

and the general solution is

$$y(x)=C_1e^{-3x}+C_2e^{-5x}$$

for some constants C_1 and C_2.

- **Double Roots.** For example, let's solve the equation

$$y''-10y'+25y=0.$$

The characteristic equation is

$$\lambda^2-10\lambda+25=0$$

which we factor into

$$(\lambda-5)^2=0.$$

Hence, $\lambda=5,5$ i.e., we have a *double* root. This still means that $y_1(x)=e^{5x}$ is a solution but the problem is that so far we have found only *one* solution, and we need *two*. It's actually not such a big problem as, it turns out, that multiplying by x always gives a second solution. Specifically,

$$y_2(x)=xe^{5x}$$

is a second solution and the general solution is

$$y(x)=C_1e^{5x}+C_2xe^{5x}$$

for some constants C_1 and C_2.

- **Complex Roots.** For example, let's solve the equation

$$y''+6y'+25y=0.$$

In this case the characteristic equation is

$$\lambda^2+6\lambda+25=0.$$

We won't be able to factor this one, so we solve using the Quadratic Formula,

$$\begin{aligned}\lambda &= \frac{-6\pm\sqrt{6^2-4\times 25}}{2}\\ &= \frac{-6\pm\sqrt{-64}}{2}\\ &= \frac{-6\pm 8i}{2}\\ &= -3\pm 4i.\end{aligned}$$

So, as you can see we have two complex (conjugate) roots. When this happens the two solutions will have sin and cos components. Specifically, in this case,

$$y_1(x) = e^{-3x}\cos(4x) \quad \text{and} \quad y_2(x) = e^{-3x}\sin(4x)$$

and the general solution is

$$y(x) = C_1 e^{-3x}\cos(4x) + C_2 e^{-3x}\sin(4x)$$

for some constants C_1 and C_2.

Non-homogeneous Equations

These are again, just the ones where the right-hand side is non-zero. The basic rule for solving them is to find any *particular* solution of the non-homogeneous equation, $y_p(x)$ say, and then the general solution is just

$$y(x) = y_h(x) + y_p(x)$$

where $y_h(x)$ is the general solution of the associated *homogeneous* equation.

Here are some examples of what can happen. It's generally better to first find the general solution of the homogeneous equation, as this can have an effect on how you have to "hunt" for a particular solution $y_p(x)$.

- **Constant RHS.** For example, let's solve

  $$y'' - 3y' + 2y = 5.$$

  First find the general solution to the homogeneous equation, $y_h(x)$. Just like before, the characteristic equation is $\lambda^2 - 3\lambda + 2 = 0$ which we factor into $(\lambda - 1)(\lambda - 2) = 0$ so that $\lambda = 1, 2$ and

  $$y_h(x) = C_1 e^x + C_2 e^{2x}$$

  for some constants C_1 and C_2.

  Now to find $y_p(x)$. The basic idea is try looking for a y_p of the same form as the RHS. In this case the RHS is a constant so we try to find a y_p that is a constant, i.e.,

  $$y_p(x) = A$$

  for some constant A.

  Substituting into the equation (since $y_p' = 0$ and $y_p'' = 0$) we have

  $$0 + 0 + 2A = 5$$

  so that $A = 5/2$ and

  $$y_p(x) = 5/2.$$

  Hence the general solution is

  $$\begin{aligned} y(x) &= y_h(x) + y_p(x) \\ &= C_1 e^x + C_2 e^{2x} + \frac{5}{2} \end{aligned}$$

  for some constants C_1 and C_2.

- **Polynomial RHS.** For example, let's solve

$$y'' - 3y' + 2y = x^2.$$

The homogeneous equation is the same as our first example and so

$$y_h(x) = C_1 e^x + C_2 e^{2x}$$

for some constants C_1 and C_2.

To find $y_p(x)$, we again try a y_p of the same general form as the RHS. In this case the RHS is x^2 which is a polynomial of degree 2 (a quadratic). Hence we try a y_p that is a general quadratic, i.e.,

$$y_p(x) = Ax^2 + Bx + C$$

for some constants A, B and C (which we will need to find). Calculating, we have

$$y_p = Ax^2 + Bx + C, \; y_p' = 2Ax + B, \; y_p'' = 2A$$

and so, substituting into the equation we obtain

$$2A - 3(2Ax + B) + 2(Ax^2 + Bx + C) = x^2.$$

Collecting terms, we have

$$2Ax^2 + (2B - 6A)x + (2C - 3B + 2A) = x^2$$

and then comparing coefficients of powers of x we end up with three equations

$$\begin{aligned} 2A &= 1 \\ 2B - 6A &= 0 \\ 2C - 3B + 2A &= 0 \end{aligned}$$

in the three unknowns A, B and C. We solve these (by Gaussian elimination) to obtain

$$A = \frac{1}{2}, \; B = \frac{3}{2} \text{ and } C = \frac{7}{4}.$$

It follows that

$$y_p(x) = \frac{1}{2}x^2 + \frac{3}{2}x + \frac{7}{4}$$

is a particular solution, and the general solution is

$$\begin{aligned} y(x) &= y_h(x) + y_p(x) \\ &= C_1 e^x + C_2 e^{2x} + (1/2)x^2 + (3/2)x + (7/4). \end{aligned}$$

- **Exponential RHS.** Let's now solve

$$y'' - 3y' + 2y = e^{3x}.$$

The homogeneous solution is, as before,

$$y_h(x) = C_1 e^x + C_2 e^{2x}$$

for some constants C_1 and C_2.

To find y_p, the RHS is an exponential and so we try a y_p that is an exponential of the same type, i.e.,

$$y_p(x) = Ae^{3x}$$

for some constant A (which we need to find).

Substituting into the equation, we get

$$\begin{aligned} 9Ae^{3x} - 3(3Ae^{3x}) + 2Ae^{3x} &= e^{3x} \\ (9A - 9A + 2A)e^{3x} &= e^{3x} \\ 2A &= 1 \\ A &= 1/2 \end{aligned}$$

and so

$$y_p(x) = \frac{1}{2}e^{3x}$$

is a particular solution and the general solution is

$$\begin{aligned} y(x) &= y_h(x) + y_p(x) \\ &= C_1e^x + C_2e^{2x} + (1/2)e^{3x}. \end{aligned}$$

There are of course, more complicated possibilities. If, for example, the RHS were x^2e^{3x} then you would try a y_p of the form

$$y_p(x) = (Ax^2 + Bx + C)e^{3x},$$

and so on.

- **Trigonometric RHS.** Let's solve

$$y'' - 3y' + 2y = \cos(4x).$$

As before,

$$y_h(x) = C_1e^x + C_2e^{2x}.$$

In this case, the RHS is a trig function and so we look for a y_p that is also a trig function of the same general type. Specifically, in this case, we try

$$y_p(x) = A\cos(4x) + B\sin(4x)$$

for some constants A and B (which we have to find). Calculating, we have

$$\begin{aligned} y_p &= A\cos(4x) + B\sin)4x) \\ y_p' &= -4A\sin(4x) + 4B\cos(4x) \\ y_p'' &= -16A\cos(4x) - 16B\sin(4x) \end{aligned}$$

so that when we substitute into the equation we get

$$\begin{aligned} \{-16A\cos(4x) - 16B\sin(4x)\} - 3\{-4A\sin(4x) + 4B\cos(4x)\} & \\ +2\{A\cos(4x) + B\sin(4x)\} &= \cos(4x) \\ (-16A - 12B + 2A)\cos(4x) + (-16B + 12A + 2B)\sin(4x) &= \cos(4x). \end{aligned}$$

Then, equating coefficients, we have

$$\begin{aligned} -14A - 12B &= 1 \\ -14B + 12A &= 0. \end{aligned}$$

Again, we solve this system of two equations in two unknowns (by Gaussian elimination) to find that

$$A = -\frac{7}{170} \quad \text{and} \quad B = -\frac{3}{85}.$$

Hence,

$$y_p(x) = -\frac{7}{170}\cos(4x) - \frac{3}{85}\sin(4x)$$

is a particular solution, and the general solution is

$$\begin{aligned} y(x) &= y_h(x) + y_p(x) \\ &= C_1 e^x + C_2 e^{2x} - (7/170)\cos(4x) - (3/85)\sin(4x) \end{aligned}$$

for some constants C_1 and C_2.

- **Mixed RHS.** It is possible that the RHS is a *combination* of the types we've been discussing. For example, what about

$$y'' - 3y' + 2y = x + e^{4x}?$$

Acutally, this is not such a problem. The way you find a y_p is to split it into two parts

$$y'' - 3y' + 2y = x \quad \text{and} \quad y'' - 3y' + 2y = e^{4x},$$

find y_p's for both of these, and then the y_p you want is just the *sum* of these two. For this particular example, for the first equation, $y'' - 3y' + 2y = x$, you try a y_p^1 of the form

$$y_p^1 = Ax + B,$$

substitute into this (first) equation as we've been doing, asdn solve for A and B to get $A = 1/2$ and $B = 3/4$, so that

$$y_p^1(x) = \frac{1}{2}x + \frac{3}{4}.$$

Similarly, for the second equation, $y'' - 3y' + 2y = e^{4x}$, you try a y_p^2 of the form

$$y_p^2 = Ae^{4x},$$

substitute into this second equation, and solve for A to get

$$A = \frac{1}{6}$$

so that

$$y_p^2(x) = \frac{1}{6}e^{4x}.$$

Hence,

$$y_p(x) = y_p^1(x) + y_p^2(x) = \frac{1}{2}x + \frac{3}{4} + \frac{1}{6}e^{4x}$$

is a particular solution and the general solution is

$$\begin{aligned} y(x) &= y_h(x) + y_p(x) \\ &= C_1 e^x + C_2 e^{2x} + \frac{1}{2}x + \frac{3}{4} + \frac{1}{6}e^{4x}, \end{aligned}$$

for some constants C_1 and C_2.

Alternatively, if you wanted to do everything together at once, you could try a single y_p of the form

$$y_p = Ax + B + Ce^{4x}.$$

There are some slight complications that could arise. Here's an example. Suppose we wanted to solve the equation

$$y'' - 3y' + 2y = e^x.$$

This looks very much like the ones we've been doing, and it is. There is however, one subtle difference. Let's see what this is.

If we followed our normal procedure, we would try a $y_p(x)$ of the form

$$y_p(x) = Ae^x.$$

But then, when we substitute into the equation, we get

$$\begin{aligned} Ae^x - 3Ae^x + 2Ae^x &= e^x \\ (A - 3A + 2A)e^x &= e^x \\ (A - 3A + 2A) &= 1 \\ 0 &= 1 \end{aligned}$$

which is *not* possible!

What went wrong? Well, the basic problem is that the RHS, e^x, is a *solution* of the homogeneous equation (recall the two roots were $\lambda = 1, 2$), and then so is Ae^x for any multiple A. Hence, when we substitute our candidate y_p into the equation, the LHS evaluates to zero, while the RHS does not! How do you get around this? It's not so bad, as all you have to do is multiply by x, i.e., try a y_p of the form

$$y_p(x) = Axe^x.$$

Let's see what happens. Calculating the drivatives, we have

$$\begin{aligned} y_p &= Axe^x \\ y_p' &= A\{e^x + xe^x\} \\ y_p'' &= A\{2e^x + xe^x\} \end{aligned}$$

so that, substituting into the equation, we obtain

$$\begin{aligned} Ae^x(2+x) - 3Ae^x(1+x) + 2Axe^x &= e^x \\ e^x\left\{A(2+x) - 3A(1+x) + 2Ax\right\} &= e^x \\ A(2+x) - 3A(1+x) + 2Ax &= 1 \\ (A - 3A + 2A)x + (2A - 3A) &= 1 \\ 0x - A &= 1 \end{aligned}$$

so that $A = -1$ and

$$y_p(x) = -xe^x$$

is a particular solution.

The general solution is then

$$\begin{aligned} y(x) &= y_h(x) + y_p(x) \\ &= C_1e^x + C_2e^{2x} - xe^x \end{aligned}$$

for some constants C_1 and C_2.

It can actually be a bit worse. Let's try the equation

$$y'' - 2y' + y = e^x.$$

We first solve the homogeneous equation. The characteristic equation is

$$\lambda^2 - 2\lambda + 1 = 0$$

which we factor into

$$(\lambda - 1)^2 = 0.$$

hence $\lambda = 1$ is a *double* root. This means that both $y_1(x) = e^x$ and $y_2(x) = xe^x$ are solutions of the homogeneous equation. Just as in the preceeding example, y_p of the form $y_p(x) = Ae^x$ is a solution of the homogeneous equation and hence no good. But now also, the fix that worked last time, multiplying by x to try $y_p = Axe^x$ is *also* a solution of the homogeneous equation, and so equally bad! What to do? Well since multiplying by x once, worked before, let's multiply by x a second time, and try

$$y_p(x) = Ax^2e^x$$

and see what happens.

Calculating the derivatives, we get

$$\begin{aligned} y_p &= Ax^2e^x \\ y_p' &= A(2xe^x + x^2e^x) \\ y_p'' &= A(2e^x + 4xe^x + x^2e^x) \end{aligned}$$

so that when we substitute into the equation we get,

$$\begin{aligned} Ae^x(2 + 4x + x^2) - 2Ae^x(2x + x^2) + Ae^x(x^2) &= e^x \\ A(2 + 4x + x^2) - 2A(2x + x^2) + Ax^2 &= 1 \\ (A - 2A + A)x^2 + (4A - 4A)x + 2A &= 1 \\ 0x^2 + 0x + 2A &= 1 \\ A &= 1/2. \end{aligned}$$

Hence,

$$y_p(x) = \frac{1}{2}x^2e^x$$

is a particular solution, and the general solution is

$$\begin{aligned} y(x) &= y_h(x) + y_p(x) \\ &= C_1e^x + C_2xe^x + (1/2)x^2e^x \end{aligned}$$

for some constants C_1 and C_2.

By the way, this procedure for finding a y_p of a certain form has a fancy name: **The Method of Undetermined Coefficients**.