MATHEMATICS 271 L01 FALL 2004
ASSIGNMENT 1 SOLUTION

1. In this question, prove your answers using **only** the definitions of rational and irrational numbers and the fact that $\sqrt{2}$ is irrational. Let $\mathcal{P}$ be the statement : "For all real numbers x and y, if $x+y$ is rational and $x-y$ is irrational then x is irrational and y is irrational."

(a) Is $\mathcal{P}$ true? Prove your answer.

(b) State the *converse* of $\mathcal{P}$. Is the *converse* of $\mathcal{P}$ true? Prove your answer.

(c) State the the *contrapositive* of $\mathcal{P}$. Is the *contrapositive* of $\mathcal{P}$ true? Explain.

(d) State the the *inverse* of $\mathcal{P}$. Is the *inverse* of $\mathcal{P}$ true? Explain.

(e) State the the *negation* of $\mathcal{P}$. Is the *negation* of $\mathcal{P}$ true? Explain.

Solution:
(a) $\mathcal{P}$ is true and here is a proof.
Proof: Let $x, y \in \mathbb{R}$ so that $x+y$ is rational and $x-y$ is irrational. Since $x+y$ is rational, there are $m, n \in \mathbb{Z}$ so that $x+y = \dfrac{m}{n}$ and $n \neq 0$.

First, we show that x is irrational by a contradiction proof. Suppose that x is rational. Then $x = \dfrac{s}{t}$ for some $s, t \in \mathbb{Z}$ where $t \neq 0$. It follows that, $x - y = 2x - (x+y) = 2\dfrac{s}{t} - \dfrac{m}{n} = \dfrac{2sn - mt}{nt}$ which implies that $x-y$ is rational (note that $2sn - mt$ and nt are integers because $2, m, n, s, t \in \mathbb{Z}$, and $nt \neq 0$ because $n \neq 0$ and $t \neq 0$). This contradicts the assumption that $x-y$ is irrational. Thus, x is irrational.

Similarly, we show that y is irrational by a contradiction proof. Suppose that y is rational. Then $x = \dfrac{p}{q}$ for some $p, q \in \mathbb{Z}$ where $q \neq 0$. It follows that, $x-y = (x+y) - 2y = \dfrac{m}{n} - 2\dfrac{p}{q} = \dfrac{mq - 2pn}{nq}$ which implies that $x-y$ is rational (note that $mq - 2pn$ and nq are integers because $2, m, n, p, q \in \mathbb{Z}$, and $nq \neq 0$ because $n \neq 0$ and $q \neq 0$). This contradicts the assumption that $x-y$ is irrational. Thus, y is irrational.

(b) The *converse* of $\mathcal{P}$ is: "For all real numbers x and y, if x is irrational and y is irrational then $x+y$ is rational and $x-y$ is irrational."

The *converse* of $\mathcal{P}$ is false. For example, when $x = y = \sqrt{2}$, we see that x is irrational and y is irrational. However, $x - y = \sqrt{2} - \sqrt{2} = 0$, which is not rational.

(c) The *contrapositive* of $\mathcal{P}$ is: "For all real numbers x and y, if x is rational or y is rational then $x+y$ is irrational or $x-y$ is rational."

The *contrapositive* of $\mathcal{P}$ is true because it is logically equivalent to $\mathcal{P}$ which is true as proven in (a).

(d) The *inverse* of $\mathcal{P}$ is "For all real numbers x and y, if $x+y$ is irrational or $x-y$ is rational then x is rational or y is rational."

The *inverse* of $\mathcal{P}$ is false because it is logically equivalent to the *converse* of $\mathcal{P}$ (we note that the inverse of $\mathcal{P}$ is the contrapositive of the converse of $\mathcal{P}$) which is false as seen in (a).

(e)The *negation* of $\mathcal{P}$ is: "There real numbers x and y so that $x+y$ is rational and $x-y$ is irrational, but x is rational or y is rational."

The *negation* of $\mathcal{P}$ false because $\mathcal{P}$ is true as proven in (a).

2. In this question, a, b and c are integers. Let $\mathcal{Q}$ be the statement : "For all integers a, b and c, if $a \mid b$ and $a \mid c$ then $a \mid 2b+c$ and $a \mid 3b+2c$.".

(a) Is $\mathcal{Q}$ true? Prove your answer.

(b) State the *converse* of $\mathcal{Q}$. Is the *converse* of $\mathcal{Q}$ true? Prove your answer.

(c) State the the *contrapositive* of $\mathcal{Q}$. Is the *contrapositive* of $\mathcal{Q}$ true? Explain.

(d) State the the *inverse* of $\mathcal{Q}$. Is the *inverse* of $\mathcal{Q}$ true? Explain.

(e) State the the *negation* of $\mathcal{Q}$. Is the *negation* of $\mathcal{Q}$ true? Explain.

Solution:

(a) $\mathcal{Q}$ is true and here is a proof.

Let a, b and c be integers so that $a \mid b$ and $a \mid c$. Since $a \mid b$ and $a \mid c$, there are $x, y \in \mathbb{Z}$ so that $b = ax$ and $c = ay$. Thus, $2b + c = 2ax + ay = a(2x + y)$ and $3b + 2c = 3ax + 2ay = a(3x + 2y)$, which imply $a \mid 2b + c$ and $a \mid 3b + 2c$ (We note that $2x + y$ and $3x + 2y$ are integers because $2, 3, x, y \in \mathbb{Z}$).

(b) The *converse* of $\mathcal{Q}$ is: "For all integers a, b and c, if $a \mid 2b+c$ and $a \mid 3b+2c$ then $a \mid b$ and $a \mid c$.".

The *converse* of $\mathcal{Q}$ is true and here is a proof.

Let a, b and c be integers so that $a \mid 2b+c$ and $a \mid 3b+2c$. Since $a \mid 2b+c$ and $a \mid 3b+2c$, there are $m, n \in \mathbb{Z}$ so that $2b + c = am$ and $3b + 2c = an$. Now, $b = 2(2b + c) - 3b + 2c = 2am - an = a(2m - n)$ and $c = 2(3b + 2c) - 3(2b + c) = 2an - 3am = a(2n - 3m)$, which imply $a \mid b$ and $a \mid c$ (We note that $2m-n$ and $2n-3m$ are integers because $2, 3, m, n \in \mathbb{Z}$).

(c) The *contrapositive* of $\mathcal{Q}$ is: "For all integers a, b and c, if $a \nmid 2b + c$ or $|\nmid 3b + 2c$ then $a \nmid b$ or $a \nmid c$."

The *contrapositive* of $\mathcal{Q}$ is true because it is logically equivalent to $\mathcal{Q}$ which is true as proven in (a).

(d) The *inverse* of $\mathcal{Q}$ is "For all integers a, b and c, if $a \nmid b$ or $a \nmid c$ then $a \nmid 2b + c$ or $a \nmid 3b + 2c$."

The *inverse* of $\mathcal{Q}$ is true because it is logically equivalent to the *converse* of $\mathcal{Q}$ (we note that the inverse of $\mathcal{Q}$ is the contrapositive of the converse of $\mathcal{Q}$) which is true as seen in (a).

(e)The *negation* of $\mathcal{Q}$ is: "There are integers a, b and c so that $a \mid b$ and $a \mid c$, but $a \nmid 2b+c$ or $a \nmid 3b + 2c$."

The *negation* of $\mathcal{Q}$ false because $\mathcal{Q}$ is true as proven in (a).

3. For each of the following statements, determine whether the statement is true or false and **prove your answer**.

(a) For all integers y, there is an integer x so that $x^3 + x = y$.

(b) For all integers x and y, if $2x^2 + x = 2y^2 + y$ then $x = y$..

(c) For all integers m and n, if $m \mid n$ and $n \mid m$ then $n = m$.

(d) For all rational number x, there is a positive integer k so that kx is an integer.

(e) There is a positive integer k so that for all rational number x, kx is an integer.

Solution:

(a) This statement is false when $y = 1$. We show that for all integers x, $x^3 + x \neq 1$. Let $x \in \mathbb{Z}$. We show $x^3 + x \neq 1$ by a contradiction proof. Suppose that $x^3 + x = 1$, that is, $x(x^2+1) = 1$ which implies that $x = x^2 + 1 = 1$ or $x = x^2 + 1 = -1$ (this is because x and $x^2 + 1$ are integers). However, these are impossible because when $x = 1$, we have $x^2 + 1 = 2 \neq 1$, and $x^2 + 1$ is positive and so $x^2 + 1 \neq -1$. Thus, $x^3 + x \neq 1$.

(b) This statement is true and here is a proof.

Let $x, y \in \mathbb{Z}$ and suppose that $2x^2 + x = 2y^2 + y$, which implies $2x^2 + x - (2y^2 + y) = 0$. Equivalently, we have $2(x^2 - y^2) + x - y = 0$ and so we get,

$$(x - y)[2(x + y) + 1] = 0 \qquad (1)$$

Note that $2(x + y) + 1$ is an odd number because $x + y$ is an integer, so $2(x + y) + 1 \neq 0$ and from (1), we get that $x - y = 0$ which implies that $x = y$.

(c) This statement is false because when $m = 1$,and $n = -1 \in \mathbb{Z}$, we have $m \mid n$ and $n \mid m$ but $m \neq n$.

(d) This statement is true and here is a proof.

Let x be a rational number; that is, $x = \frac{s}{t}$ for some integers s and t where $t \neq 0$. Let $k = |t|$. Then k is a positive integer because $t \neq 0$, and $kx = |t| \frac{s}{t} = \pm s$ which is an integer.

(e) This statement is false because we can prove that for all positive integers k, there is a rational number x such that kx is not an integer. Let k be a positive integer. Consider the rational number $x = \frac{1}{2k}$, we see that $kx = k\frac{1}{2k} = \frac{1}{2}$ which is not an integer.