MATHEMATICS 271 L01 FALL 2004
ASSIGNMENT 3 SOLUTION

1. The Fibonacci sequence $f_1,\ f_2,\ f_3 \cdots$ is defined by $f_1 = f_2 = 1$ and for integers $k \geq 3$, $f_k = f_{k-1} + f_{k-2}$.

(a) Prove that $f_n < \left(\frac{7}{4}\right)^{n-1}$ for all integers $n \geq 2$.

(b) Prove that $\sum\limits_{i=1}^{n} {f_i}^2 = f_{n+1}f_n$ for all integers $n \geq 1$.

(c) Prove that $\sum\limits_{i=1}^{n} f_i = f_{n+2} - 1$ for all integers $n \geq 1$.

(d) Prove that $\gcd(f_{n+1}, f_n) = 1$ for all integers $n \geq 1$.

Solution: We prove these by induction on n.

(a) **Basis** ($n = 2, 3$)

$f_2 = 1 < \frac{7}{4} = \left(\frac{7}{4}\right)^1 = \left(\frac{7}{4}\right)^{2-1}$ and

$f_3 = f_2 + f_1 = 1 + 1 = 2 = \frac{32}{16} < \frac{49}{16} = \left(\frac{7}{4}\right)^2 = \left(\frac{7}{4}\right)^{3-1}$

Inductive Step: Let $k \geq 4$ be an integer and suppose that

$f_m < \left(\frac{7}{4}\right)^{m-1}$ for all integers m where $2 \leq m < k$. [IH]

We want to show that $f_k < \left(\frac{7}{4}\right)^{k-1}$.

Now, since $k \geq 4$, by the definition of The Fibonacci sequence we have

$$\begin{aligned}
f_k &= f_{k-1} + f_{k-2} \\
&< \left(\tfrac{7}{4}\right)^{k-2} + \left(\tfrac{7}{4}\right)^{k-3} && \text{because } 2 \leq k-1, k-2 < k \text{ and by [IH]} \\
&= \left(\tfrac{7}{4}\right)^{k-1}\left(\left(\tfrac{7}{4}\right)^{-1} + \left(\tfrac{7}{4}\right)^{-2}\right) \\
&= \left(\tfrac{7}{4}\right)^{k-1}\left(\tfrac{4}{7} + \tfrac{16}{49}\right) \\
&= \left(\tfrac{7}{4}\right)^{k-1}\left(\tfrac{4}{7} + \tfrac{16}{49}\right) \\
&= \left(\tfrac{7}{4}\right)^{k-1}\left(\tfrac{44}{49}\right) \\
&< \left(\tfrac{7}{4}\right)^{k-1} && \text{because } \tfrac{44}{49} < 1
\end{aligned}$$

Thus, we have prove by induction (strong form) that $f_n < \left(\frac{7}{4}\right)^{n-1}$ for all integers $n \geq 2$.

(b) **Basis** ($n = 1$)

$\sum\limits_{i=1}^{1} {f_i}^2 = {f_1}^2 = 1^2 = 1 = 1 \times 1 = f_2 f_1 = f_{1+1} f_1$

Inductive Step: Let $k \geq 1$ be an integer and suppose that

$\sum\limits_{i=1}^{k} {f_i}^2 = f_{k+1}f_k$. [IH]

We want to show that $\sum\limits_{i=1}^{k+1} f_{ik}^2 = f_{k+2}f_{k+1}$.

Now,

$$\begin{aligned}\sum_{i=1}^{k+1} {f_i}^2 &= \left(\sum_{i=1}^{k} {f_i}^2\right) + f_{k+1}^2 \\ &= f_{k+1}f_k + f_{k+1}^2 && \text{by [IH]} \\ &= f_{k+1}\left(f_k + f_{k+1}\right) && \text{because } f_{k+2} = f_k + f_{k+1} \\ &= f_{k+1}f_{k+2} \\ &= f_{k+2}f_{k+1}\end{aligned}$$

Thus, we have prove by induction that $\sum_{i=1}^{n} {f_i}^2 = f_{n+1}f_n$ for all integers $n \geq 1$.

(c) **Basis** $(n = 1)$

$\sum_{i=1}^{1} f_i = 1 = 2 - 1 = f_3 - 1 = f_{1+2} - 1.$

Inductive Step: Let $k \geq 1$ be an integer and suppose that

$\sum_{i=1}^{k} f_i = f_{k+2} - 1.$ [IH]

We want to show that $\sum_{i=1}^{k+1} f_i = f_{k+3} - 1.$

Now,

$$\begin{aligned}\sum_{i=1}^{k+1} f_i &= \left(\sum_{i=1}^{k} f_i\right) + f_{k+1} \\ &= f_{k+2} - 1 + f_{k+1} && \text{by [IH]} \\ &= f_{k+2} + f_{k+1} - 1 \\ &= f_{k+3} - 1. && \text{because } f_{k+3} = f_{k+2} + f_{k+1}\end{aligned}$$

Thus, we have prove by induction that $\sum_{i=1}^{n} f_i = f_{n+2} - 1$ for all integers $n \geq 1$.

(d) **Basis** $(n = 1)$

$\gcd\left(f_{1+1}, f_1\right) = \gcd\left(f_2, f_1\right) = \gcd\left(1, 1\right) = 1$

Inductive Step: Let $k \geq 1$ be an integer and suppose that

$\gcd\left(f_{k+1}, f_k\right) = 1.$ [IH]

We want to show that $\gcd\left(f_{k+2}, f_{k+1}\right) = 1$.

Now, since $k + 2 \geq 3$, by the definition of The Fibonacci sequence we have $f_{k+2} = f_{k+1} + f_k = f_{k+1} \times 1 + f_k$ and so by Lemma 3.8.2, we have $\gcd\left(f_{k+2}, f_{k+1}\right) = \gcd\left(f_{k+1}, f_k\right) = 1$.

Thus, we have prove by induction that $\gcd\left(f_{n+1}, f_n\right) = 1$ for all integers $n \geq 1$.

2. Prove the following statements:

(a) $\sum_{i=1}^{n} \frac{1}{i^2} < 2 - \frac{1}{n}$ for all integers $n \geq 2$.

(b) $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} > 2\left(\sqrt{n+1} - 1\right)$ for all integers $n \geq 1$.

(c) $\dfrac{1 \cdot 3 \cdot 5 \cdots (2n-3) \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n-2) \cdot (2n)} \geq \dfrac{1}{2n}$ for all integers $n \geq 1$.

(d) $\dfrac{1 \cdot 3 \cdot 5 \cdots (2n-3) \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n-2) \cdot (2n)} \leq \dfrac{1}{\sqrt{n+1}}$ for all integers $n \geq 1$.

Solution: We prove these by induction on n.

(a) **Basis** ($n = 2$)

$\sum_{i=1}^{2} \frac{1}{i^2} = \frac{1}{1^2} + \frac{1}{2^2} = 1 + \frac{1}{4} = \frac{3}{4} < \frac{6}{4} = \frac{3}{2} = 2 - \frac{1}{2}$

Inductive Step: Let $k \geq 2$ be an integer and suppose that

$\sum_{i=1}^{k} \frac{1}{i^2} < 2 - \frac{1}{k}$ [IH]

We want to show that $\sum_{i=1}^{k+1} \frac{1}{i^2} < 2 - \frac{1}{k+1}$

Now,

$$\begin{aligned}
\sum_{i=1}^{k+1} \frac{1}{i^2} &= \left(\sum_{i=1}^{k} \frac{1}{i^2}\right) + \frac{1}{(k+1)^2} \\
&< 2 - \frac{1}{k} + \frac{1}{(k+1)^2} && \text{by [IH]} \\
&= 2 - \left(\frac{1}{k} - \frac{1}{(k+1)^2}\right) \\
&= 2 - \frac{(k+1)^2 - k}{k(k+1)^2} \\
&= 2 - \frac{k^2+2k+1-k}{k(k+1)^2} \\
&= 2 - \frac{k^2+k+1}{k(k+1)^2} \\
&< 2 - \frac{k^2+k}{k(k+1)^2} && \text{because } k^2+k+1 > k^2+k \\
&= 2 - \frac{k(k+1)}{k(k+1)^2} \\
&= 2 - \frac{1}{k+1}
\end{aligned}$$

Thus, we have prove by induction that $\sum_{i=1}^{n} \frac{1}{i^2} < 2 - \frac{1}{n}$ for all integers $n \geq 2$.

(b) **Basis** ($n = 1$)

We note that $1.5 > \sqrt{2}$ because $(1.5)^2 = 2.25 > 2 = \left(\sqrt{2}\right)^2$, and so $\sum_{i=1}^{1} \frac{1}{\sqrt{i}} = \frac{1}{\sqrt{1}} = 1 = 2\,(1.5-1) > 2\left(\sqrt{2}-1\right) = 2\left(\sqrt{1+1}-1\right)$.

Inductive Step: Let $k \geq 1$ be an integer and suppose that

$\sum_{i=1}^{k} \frac{1}{\sqrt{i}} > 2\left(\sqrt{k+1}-1\right)$ [IH]

We want to show that $\sum_{i=1}^{k+1} \frac{1}{\sqrt{i}} > 2\left(\sqrt{k+2}-1\right)$.

Now,

$$\begin{aligned}
\sum_{i=1}^{k+1} \tfrac{1}{\sqrt{i}} &= \left(\sum_{i=1}^{k} \tfrac{1}{\sqrt{i}}\right) + \tfrac{1}{\sqrt{k+1}} \\
&> 2\left(\sqrt{k+1}-1\right) + \tfrac{1}{\sqrt{k+1}} && \text{by [IH]} \\
&= 2\sqrt{k+1} + \tfrac{1}{\sqrt{k+1}} - 2 \\
&= \tfrac{2(k+1)+1}{\sqrt{k+1}} - 2 \\
&= \tfrac{2k+3}{\sqrt{k+1}} - 2 \\
&= \tfrac{2(2k+3)\sqrt{k+2}}{2\sqrt{k+1}\sqrt{k+2}} - 2 \\
&= 2\sqrt{k+2}\tfrac{\sqrt{(2k+3)^2}}{2\sqrt{k+1}\sqrt{k+2}} - 2 \\
&= 2\sqrt{k+2}\sqrt{\tfrac{4k^2+12k+9}{4(k+1)(k+2)}} - 2 \\
&= 2\sqrt{k+2}\sqrt{\tfrac{4k^2+12k+9}{4k^2+12k+8}} - 2 \\
&> 2\sqrt{k+2}\sqrt{1} - 2 && \text{because } \tfrac{4k^2+12k+9}{4k^2+12k+8} > 1 \\
&= 2\sqrt{k+2} - 2 \\
&= 2\left(\sqrt{k+2}-1\right).
\end{aligned}$$

Thus, we have prove by induction that $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} > 2\left(\sqrt{n+1}-1\right)$ for all integers $n \geq 1$.

(c) **Basis** ($n = 1$)

$$\frac{1}{2} \geq \frac{1}{2} = \frac{1}{2 \times 1}$$

Inductive Step: Let $k \geq 1$ be an integer and suppose that

$$\frac{1 \cdot 3 \cdot 5 \cdots (2k-3) \cdot (2k-1)}{2 \cdot 4 \cdot 6 \cdots (2k-2) \cdot (2k)} \geq \frac{1}{2k} \qquad \text{[IH]}$$

We want to show that $\dfrac{1 \cdot 3 \cdot 5 \cdots (2k-1) \cdot (2k+1)}{2 \cdot 4 \cdot 6 \cdots (2k) \cdot (2k+2)} \geq \dfrac{1}{2(k+1)}$

Now,

$$\begin{aligned}
\frac{1 \cdot 3 \cdot 5 \cdots (2k-1) \cdot (2k+1)}{2 \cdot 4 \cdot 6 \cdots (2k) \cdot (2k+2)} &= \frac{1 \cdot 3 \cdot 5 \cdots (2k-3) \cdot (2k-1)}{2 \cdot 4 \cdot 6 \cdots (2k-2) \cdot (2k)} \cdot \frac{(2k+1)}{(2k+2)} \\
&\geq \frac{1}{2k} \cdot \frac{(2k+1)}{(2k+2)} && \text{by [IH]} \\
&= \frac{2k+1}{2k} \cdot \frac{1}{(2k+2)} \\
&> 1 \cdot \frac{1}{(2k+2)} && \text{because } 2k+1 > 2k \\
&= \frac{1}{(2k+2)} = \frac{1}{2(k+1)}
\end{aligned}$$

Thus, we have prove by induction that $\dfrac{1 \cdot 3 \cdot 5 \cdots (2n-3) \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n-2) \cdot (2n)} \geq \dfrac{1}{2n}$ for all integers $n \geq 1$.

(d) **Basis** ($n = 1$) Since $\sqrt{2} < 2$, we have

$$\frac{1}{2} < \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{1+1}}$$

Inductive Step: Let $k \geq 1$ be an integer and suppose that

$$\frac{1 \cdot 3 \cdot 5 \cdots (2k-3) \cdot (2k-1)}{2 \cdot 4 \cdot 6 \cdots (2k-2) \cdot (2k)} \leq \frac{1}{\sqrt{k+1}} \qquad \text{[IH]}$$

We want to show that $\dfrac{1 \cdot 3 \cdot 5 \cdots (2k-1) \cdot (2k+1)}{2 \cdot 4 \cdot 6 \cdots (2k) \cdot (2k+2)} \leq \dfrac{1}{\sqrt{k+2}}$

Now,

$$\begin{aligned}
\frac{1 \cdot 3 \cdot 5 \cdots (2k-1) \cdot (2k+1)}{2 \cdot 4 \cdot 6 \cdots (2k) \cdot (2k+2)} &= \frac{1 \cdot 3 \cdot 5 \cdots (2k-3) \cdot (2k-1)}{2 \cdot 4 \cdot 6 \cdots (2k-2) \cdot (2k)} \cdot \frac{(2k+1)}{(2k+2)} \\
&\leq \frac{1}{\sqrt{k+1}} \cdot \frac{(2k+1)}{(2k+2)} && \text{b} \\
&= \frac{(2k+1)}{2(k+1)\sqrt{k+1}} \\
&= \frac{2k+1}{2\sqrt{(k+1)^3}} \\
&= \frac{2k+1}{2\sqrt{k^3+3k^2+3k+1}} \\
&= \frac{2k+1}{\sqrt{4k^3+12k^2+12k+4}} \\
&< \frac{2k+1}{\sqrt{4k^3+12k^2+9k+2}} && \text{because } 4k^3+12k^2+12 \\
&= \frac{2k+1}{\sqrt{(k+2)(4k^2+4k+1)}} \\
&= \frac{2k+1}{\sqrt{(k+2)(2k+1)^2}} \\
&= \frac{2k+1}{(2k+1)\sqrt{k+2}} \\
&= \frac{1}{\sqrt{k+2}}
\end{aligned}$$

Thus, we have prove by induction that $\dfrac{1 \cdot 3 \cdot 5 \cdots (2n-3) \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n-2) \cdot (2n)} \leq \dfrac{1}{\sqrt{n+1}}$ for all integers $n \geq 1$.

3. Prove or disprove the following statements:

(a) For all sets A, B and C, if $A \cup B = A \cup C$ then $B = C$.

(b) For all sets A, B and C, if $A = B \cup C$ then $A - B = C$.

(c) For all sets A, B and C, if $A - B = C$ then $A = B \cup C$.

(d) For all sets A, B and C, if $A - (B \cap C) = \emptyset$ then $A - C \subseteq B$.

(e) For all sets A, B and C, if $A - C \subseteq B$ then $A - (B \cap C) = \emptyset$.

Solution:

(a) This statement is false. For example, when $A = B = \{1\}$ and $C = \emptyset$, we have $A \cup B = \{1\} \cup \{1\} = \{1\} = \{1\} \cup \emptyset = A \cup C$, but $B = \{1\} \neq \emptyset = C$.

(b) This statement is false. For example, when $A = B = C = \{1\}$, we have
$A = \{1\} = \{1\} \cup \{1\} = B \cup C$, but $A - B = \{1\} - \{1\} = \emptyset \neq \{1\} = C$.
(c) This statement is false. For example, when $A = C = \emptyset$ and $B = \{1\}$, we have
$A - B = \emptyset - \{1\} = \emptyset = C$, but $A = \emptyset \neq \{1\} = \{1\} \cup \emptyset = B \cup C$.
(d) This statement is true and here is a proof. Let A, B, C be sets so that $A-(B \cap C) = \emptyset$. We show that $A - C = \emptyset$ by a contradiction proof. Suppose that $A - C \neq \emptyset$, that is, there exist $x \in A-C$. Since $x \in A-C$, $x \in A$ and $x \notin C$. Since $x \notin C$, $x \notin B \cap C$. Since $x \in A$ and $x \notin B \cap C$, $x \in A - (B \cap C)$ which contradicts the assumption that $A - (B \cap C) = \emptyset$. Thus, $A - C = \emptyset \subseteq B$.
(c) This statement is false. For example, when $A = C = \{1\}$ and $B = \emptyset$, we have
$A - C = \{1\} - \{1\} = \emptyset \subseteq B$, but $A - (B \cap C) = \{1\} - (\emptyset \cap \{1\}) = \{1\} - \emptyset = \{1\} \neq \emptyset$.