MATHEMATICS 271 L01 FALL 2004
ASSIGNMENT 5

Due at 11:00 am on Friday, November 26. Your assignment must be handed in at the beginning of the lecture on November 26. Assignment must be understandable to the marker (i.e., logically correct as well as legible), and must be done by the student in his / her own words. Answer all questions, but only one question per assignment will be marked for credit. Please make sure that: (i) the cover page has **only** your student ID number, (ii) your name and ID number are on the top right corners of **all** the remaining pages, and (iii) **STAPLE** your papers.

1. Let A and B be nonempty sets. Let $f : A \to B$ be a function. Let $\mathcal{S}$ be a relation on B. Let $\mathcal{R}$ be the relation on A defined by:

$(a, b) \in \mathcal{R}$ *if and only if* $(f(a), f(b)) \in \mathcal{S}$.

Let $\mathcal{A}$ be the statement: "if $\mathcal{S}$ is an equivalence relation on B then $\mathcal{R}$ is an equivalence relation on A."

(a) Prove that $\mathcal{A}$ is true.

(b) Prove that the converse of $\mathcal{A}$ is false.

(c) Prove that if f is onto B then the converse of $\mathcal{A}$ is true.

(d) Now, suppose that $A = B = \mathbb{Z}^+$, the set of all positive integers, and $\mathcal{S}$ is the relation "congruence modulo 5". Then, by part (a), $\mathcal{R}$ is an equivalence relation on $A = \mathbb{Z}^+$. Find a one-to-one function $f : A \to B$ so that $\mathcal{R}$ has exactly two equivalence classes

2. Let a and n be integers with $n \geq 2$. Define the relation $\mathcal{R}_{a,n}$ on $\mathbb{Z}$ by $(x, y) \in \mathcal{R}_{a,n}$ if and only if $ax \equiv ay \pmod{n}$.

(a) Prove that $\mathcal{R}_{a,n}$ is an equivalence relation on $\mathbb{Z}$.

(b) Show that for all integers $n \geq 2$, there is an integer $a \neq 0$ so that $\mathcal{R}_{a,n}$ has only one equivalence class.

(c) Show that there are integers a and n, where $n \geq 2$, so that $\mathcal{R}_{a,n}$ has exactly two equivalence classes.

(d) Suppose that a and n are relatively prime. Prove that $(x, y) \in \mathcal{R}_{a,n}$ if and only if $x \equiv y \pmod{n}$.

(e) Prove that if a and n are relatively prime then $\mathcal{R}_{a,n}$ has n equivalence classes.

3. Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Give reasons to support your answers.

(a) Is it true that there is a relation $\mathcal{R}$ on A so that $\mathcal{R}$ is not symmetric, not antisymmetric but $\mathcal{R}$ is transitive?

(b) Is it true that there is a relation $\mathcal{S}$ on A so that $\mathcal{S}$ is symmetric, antisymmetric but $\mathcal{S}$ is not transitive?

(c) How many symmetric relations on A are there?

(d) How many antisymmetric relations on A are there?

(d) How many relations on A are there that are antisymetric or symmetric?